Mathematisches Institut der LMU
Prof. P. T. Nam
D.T. Nguyen

Functional Analysis II
Winter Semester 2017/18
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# **Excercise Sheet 10** for 12. 1. 2018

**10.1.** Let $A \geq B \geq 0$ be two trace class operators. Let $f : [0, \infty) \to [0, \infty)$ be a continuous, increasing function. Prove that

$$\mathrm{Tr}[f(A)] \geq \mathrm{Tr}[f(B)].$$

Here $f(A)$ and $f(B)$ are defined by Spectral Theorem.

**10.2.** Prove the Hardy-Kato inequality

$$\langle f, \sqrt{-\Delta} f \rangle \geq \frac{2}{\pi} \int_{\mathbb{R}^3} \frac{|f(x)|^2}{|x|} \, \mathrm{d}x, \quad \forall f \in H^{1/2}(\mathbb{R}^3).$$

Hint: You can mimic the proof of Hardy's inequality.

**10.3.** (a) Let $\{u_j\}_{j=1}^N$ be an orthonormal family in $L^2(\mathbb{R}^3)$, such that $u_j \in H^1(\mathbb{R}^3)$ for all $j$. Prove the fractional Lieb-Thirring inequality

$$\sum_{j=1}^{N} \langle u_j, \sqrt{-\Delta} u_j \rangle_{L^2} \geq K \int_{\mathbb{R}^3} \rho^{4/3}, \quad \text{with} \quad \rho(x) = \sum_{j=1}^{N} |u_j(x)|^2.$$

Here $K$ is a universal constant independent of $N$ and $u_j$'s.

Hint: You can mimic Rumin's proof of the Lieb-Thirring inequality.

(b) Deduce that for every $V : \mathbb{R}^3 \to \mathbb{R}$, $V_+ \in L^\infty$ and $V_- \in L^4$,

$$\mathrm{Tr}[\sqrt{-\Delta} + V]_- \leq C \int_{\mathbb{R}^3} V_-^4.$$

Here $C$ is a universal constant independent of $V$.

**10.4.** Consider the Schrödinger operator $A = -\Delta + V$ on $L^2(\mathbb{R}^2)$ with $V \in L^1 \cap L^\infty(\mathbb{R}^2)$. For every $\lambda > 0$, let $N_\lambda$ be the number of eigenvalues $< -\lambda$ of $A$. Prove that for every $n \in \mathbb{N}$,

$$N_\lambda \leq \frac{C_n}{\lambda^{1/n}} \int_{\mathbb{R}^2} |V(x)|^{1+1/n} \, \mathrm{d}x.$$

Here the constant $C_n$ is independent of $V$ and $\lambda$.

Hint: You can mimic the proof of the CLR bound.