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Mathematical Quantum Mechanics
Winter Semester 2019/20
24.01.2020

Tutorial Sheet 9

9.1. Let $A : D(A) \to H$ be a self-adjoint operator. Assume that A is bounded from below. Let $\{B_m\}$ be a sequence of bounded self-adjoint operators such that $\lim_{m\to\infty} \|B_m\| = 0$. Prove that

$$\lim_{m\to\infty} \mu_n(A + B_m) = \mu_n(A), \quad \forall n \geq 1.$$

9.2. Let $V \in L^{3/2}(\mathbb{R}^3)$. Let $A = -\Delta + V$ be an operator on $C_c^\infty(\mathbb{R}^3)$.

(i) Prove that A is bounded from below.
(ii) Prove that $\mu_n(A) \leq 0$ and $\mu_n(A) \to 0$ as $n \to \infty$.

Hint. You can write $V = V_1 + V_2 \in L_\varepsilon^{3/2}(\mathbb{R}^3) + L^2(\mathbb{R}^3)$ *for an arbitrary* $\varepsilon > 0$ *and use the fact that* $\sigma_{\text{ess}}(-k\Delta + V_2) = [0, \infty)$ *for any* $k > 0$.

9.3. Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $U(t) = e^{-itA}$ where $t \in \mathbb{R}$.

(i) Prove that if $\lim_{t\to 0} \dfrac{U(t)\psi - \psi}{t}$ exists then $\psi \in D(A)$.
(ii) Prove that if $D(A) = \mathcal{H}$ then $\lim_{t\to 0} \|U(t) - 1\| = 0$ and U is a strongly continuous one-parameter group with infinitesimal generator A.

Remark. It is also true that $D(A) = \mathcal{H}$ *if* $\lim_{t\to 0} \|U(t) - 1\| = 0$.

9.4. Let χ be a measurable real-valued function on $\mathbb{R}$. Define the transformation operator $U(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $U(t)\psi(x) = e^{-it\chi(x)}\psi(x)$ where $t \in \mathbb{R}$.

(i) Prove that U is a strongly continuous one-parameter unitary group.
(ii) Prove that the operator A defined by

$$A\psi(x) = \chi(x)\psi(x) \quad \text{with} \quad D(A) = \left\{ \psi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |\chi(x)\psi(x)|^2 \mathrm{d}x < \infty \right\}$$

is the infinitesimal generator of U.

Hint. You can define $A_n := \left\{ x \in \mathbb{R} : |\chi(x)| \geq \frac{1}{\sqrt{|t_n|}} \right\}$ *where* $t_n \to 0$ *as* $n \to \infty$ *and use a result in HW 3.1 to obtain that* $\mu(A_n) \to 0$ *as* $n \to \infty$.