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Mathematical Quantum Mechanics
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Tutorial Sheet 8

8.1. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $H^s(\mathbb{R}^d)$ where $s > 0$ and $d \geq 1$. Assume that $f_n(x) \to f(x)$ for a.e. $x \in \mathbb{R}^d$. Prove that $f \in H^s(\mathbb{R}^d)$.

8.2. Define the operator $A : D(A) \to L^2(\mathbb{R})$ by

$$(Au)(x) = -u''(x) + |x|u(x), \quad \forall u \in D(A) = C_c^\infty(\mathbb{R}).$$

Prove that $A \geq c_0$ for a constant $c_0 > 0$.

Hint. You can write $u = u\mathbb{1}(|x| \leq \epsilon) + u\mathbb{1}(|x| \geq \epsilon)$ *and use Sobolev's inequality* $\|u\|_{L^\infty(\mathbb{R})} \leq C\|u\|_{H^1(\mathbb{R})}$*, for some constant* $C > 0$*, which has been proved in the lecture.*

8.3. Let $V \in L^p(\mathbb{R}^d)$ where $d \geq 3$ and $d/2 < p \leq \infty$. Let $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^d)$ such that $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^d)$. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} V(x)|u_n(x)|^2 \mathrm{d}x = 0.$$

Hint. You can write $V = V\mathbb{1}(|x| \leq R) + V\mathbb{1}(|x| \geq R)$ *and use Sobolev's inequality.*

8.4. Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator and let $u \in \mathcal{H}$ be a normalized vector.

(i) Determine eigenvalues of the operator $P = |u\rangle\langle u|$.
(ii) Prove that if $A \geq 0$ then $A - P$ has at most one negative eigenvalue.