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Mathematical Quantum Mechanics
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Tutorial Sheet 7

7.1. Let $A : D(A) \to \mathcal{H}$ and $B : D(B) \to \mathcal{H}$ be two self-adjoint operators and bounded from below. Prove that if $A \geq B$ then $\mu_n(A) \geq \mu_n(B)$ for all $n \geq 1$.

Remark. *$A \geq B$ is defined by $D(A) \subset D(B)$ and $\langle u, Au \rangle \geq \langle u, Bu \rangle$ for all $u \in D(A)$.*

7.2. Let $A : D(A) \to \mathcal{H}$ and $B : D(B) \to \mathcal{H}$ be two bounded self-adjoint operators and bounded from below.

(i) Prove that for all $n \geq 1$ and for any constant C we have

$$\mu_n(A + C) = \mu_n(A) + C.$$

(ii) Prove that for all $n \geq 1$ we have

$$|\mu_n(A) - \mu_n(B)| \leq \|A - B\|.$$

7.3. Prove that for any $u \in L^1 \cap L^2(\mathbb{R}^d)$ we have

$$\lim_{|t|\to\infty} \left\| e^{it\Delta}u(x) - \frac{e^{\frac{i|x|^2}{4t}}}{(4\pi it)^{\frac{d}{2}}} \hat{u}\left(\frac{x}{4\pi t}\right) \right\|_{L^2} = 0.$$

Remark. *$e^{it\Delta}$ is the heat operator defined by* $e^{it\Delta}u(x) = \displaystyle\int_{\mathbb{R}^d} \frac{e^{\frac{i|x-y|^2}{4t}}}{(4\pi it)^{\frac{d}{2}}} u(y)\mathrm{d}y.$

7.4. Let $\lambda > 0$. Defined the operator $U_\lambda : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$(U_\lambda \psi)(x) := \lambda^{\frac{d}{2}} \psi(\lambda x).$$

(i) Prove that U_λ is unitary, $U_\lambda^{-1} = U_{\lambda^{-1}}$ and U_λ leaves $H^m(\mathbb{R}^d)$ invariant for any $m \in \mathbb{N}$.

(ii) Prove that the following holds on $H^2(\mathbb{R}^d)$

$$U_\lambda^{-1} \Delta U_\lambda = \lambda^2 \Delta \quad \text{and} \quad U_\lambda^{-1} V(x) U_\lambda = V(\lambda^{-1}x)$$

with V being multiplication operator satisfying $VH^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$.