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Mathematical Quantum Mechanics
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Tutorial Sheet 6

6.1. Let $\alpha \in [0,2]$ and let $A = -\Delta - |x|^{-\alpha}$ on $L^2(\mathbb{R}^3)$ with $D(A) = C_c^2(\mathbb{R}^3)$.

(i) Prove by Hölder's inequality and Hardy's inequality that

$$\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^\alpha} \mathrm{d}x \le 2^\alpha \|\nabla u\|_{L^2}^\alpha \|u\|_{L^2}^{2-\alpha}, \quad \forall u \in C_c^2(\mathbb{R}^3).$$

(ii) Prove that A is bounded from below and the quadratic form domain of A is $H^1(\mathbb{R}^3)$.

Remark. If $A \ge -C$ then the quadratic form associated to A is defined by $Q_A(x,y) = \langle x, (A+C+1)y\rangle$ for $x, y \in D(A)$.

6.2. Let $A : D(A) \to \mathcal{H}$ be a bounded operator and let $B : D(B) \to \mathcal{H}$ be a compact operator. Prove that AB and BA are compact operators.

6.3. Let $A : D(A) \to \mathcal{H}$ be a self-adjoint positive operator and $B : D(B) \to \mathcal{H}$ be a symmetric operator. Prove that B is A-compact, i.e., $B(A+i)^{-1}$ is compact if and only if $B(A+1)^{-1}$ is compact.

6.4. Let $A : D(A) \to H$ and $B : D(B) \to H$ be two self-adjoint operators.

(i) Let $\lambda \in \sigma(A)$ and let $\{u_n\}_{n=1}^\infty$ be a sequence in $D(A)$ such that $\|(A-\lambda)u_n\| \to 0$ as $n \to \infty$. Prove that for any $z \in \rho(A)$ we have

$$\lim_{n\to\infty} \left\| \left((A-z)^{-1} - (\lambda - z)^{-1}\right) u_n \right\| = 0.$$

(ii) Assume that $(A-z)^{-1} - (B-z)^{-1}$ is compact for one $z \in \rho(A) \cap \rho(B)$. Prove that

$$\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B).$$

Hint. You may prove that if $\{u_n\}_{n=1}^\infty \subset D(A)$ is a Weyl sequence for (A, λ) then $\left\{\frac{v_n}{\|v_n\|}\right\}_{n=1}^\infty \subset D(B)$ is a Weyl sequence for (B, λ), where $v_n = (B-z)^{-1}u_n$.