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Mathematical Quantum Mechanics
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Tutorial Sheet 5

5.1. Let $A = -\Delta$ with $D(A) = \{f \in H^2(0,1), f(0) = f(1) = 0\}$ and let $g(x) = x(1-x)$. Find the error in the following argument: since A is symmetric, we have $4 = \langle Ag, Ag\rangle = \langle g, A^2 g\rangle = 0$.

5.2. Prove the ellipticity

$$\|u\|_{H^2(\mathbb{R}^d)} \le C(\|u\|_{L^2(\mathbb{R}^d)} + \|\Delta u\|_{L^2(\mathbb{R}^d)}), \quad \forall u \in H^2(\mathbb{R}^d),$$

where $C > 0$ is a universal constant.

5.3. An operator is called *normal* if and only if $\|Au\| = \|A^* u\|$ for all $u \in D(A) = D(A^*)$. Prove that if A is normal then so is $A + z$ for any $z \in \mathbb{C}$.

5.4. Let B be a (densely defined) closed operator.

(i) Prove that B^*B is non-negative and symmetric, where

$$D(B^*B) = \{x \in D(B) : Bx \in D(B^*)\}.$$

(ii) Prove that B^*B is self-adjoint.

Hint. You may use the fact that $A \subset A^*$ iff A is symmetric and the Lax–Milgram theorem with the quadratic form defined by

$$Q(x, y) = \langle Bx, By\rangle + \langle x, y\rangle, \quad \forall x, y \in D(B).$$