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Mathematical Quantum Mechanics
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Tutorial Sheet 4

4.1. Let A be a bounded operator on a Hilbert space. Find an example to show that

$$\|A\| = \sup_{\|u\|=1} |\langle u, Au\rangle|$$

is not true if A is not symmetric.

4.2. Let (Ω, μ) be a measure space and let a be a fixed measurable function. Assume that $a \notin L^\infty(\Omega)$. Set $E = \cup_{k=1}^\infty E_k$ where $E_k = \{x \in \Omega : k \le |a(x)| < k+1\}$. Define

$$f(x) = \begin{cases} k^{-1}\mu(E_k)^{-1/2} & \text{if } x \in E_k \\ 0 & \text{if } x \notin E \end{cases}$$

Prove that $f \in L^2(\Omega)$ but $af \notin L^2(\Omega)$.

Remark. This exercice together with the spectral theorem said that an operator is bounded if its domain is the whole Hilbert space.

4.3. Let $A : D(A) \to \mathcal{H}$ and $B : D(B) \to \mathcal{H}$ be symmetric operators such that $A \subset B$ and B is closed. Prove that

$$\overline{A} \subset B \subset A^*.$$

Remark. $x \in D(\overline{A})$ *if and only if there exist a sequence* $\{x_n\} \subset D(A)$ *and* $y \in \mathcal{H}$ *such that* $x_n \to x$ *and* $Ax_n \to y$. *In addition, we have* $\overline{A}x = y$.

4.4. Let $A : D(A) \to \mathcal{H}$ be a symmetric operator. Prove that

(i) A^{-1} is closed if A is closed and injective.
(ii) A is closed if A is bounded
(iii) A is closed if $\rho(A) \neq \emptyset$ where

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is bijective and } (A-\lambda)^{-1} \text{ is bounded}\}.$$

(iv) A^* is closed if A is densely defined.

Remark. A *is closed if and only if for any sequence* $\{x_n\} \subset D(A)$ *such that* $x_n \to x$ *and* $Ax_n \to y$ *we must have* $x \in D(A)$ *and* $y = Ax$.