P5. a) $A \geq 1$, $\langle f, g \rangle_{Q_A} = \langle f, Af \rangle$ $\forall f, g \in D(A)$.

Claim: $Q(A) = \{ f \in L^2, \sqrt{1+x^2}\, f \in L^2, \int f = 0 \} =: K$.

Proof: By def, $Q(A) = \overline{D(A)}^{\|\cdot\|_{Q_A}} = \{ f \in L^2 : \exists \{f_n\} \subset D(A),$
$f_n \to f$ wrt $\|\cdot\|_{Q_A} \}$.

$= \{ f \in L^2, \exists \{f_n\} \subset D(A), f_n \to f$ in L^2,
f_n is Cauchy sequence wrt $\|\sqrt{A}\cdot\|_{L^2} \}$

"$\subset$": let $f \in Q(A)$ then $\exists \{f_n\} \subset D(A)$ st $f_n \to f$ in L^2 and f_n is Cauchy sequence wrt $\|\sqrt{A}\cdot\|_{L^2} \Rightarrow \sqrt{1+x^2}\, f_n \to g$ in L^2. By the same argument as HW 7, 2, $\sqrt{1+x^2}\, f = g \in L^2$.

Furthermore, $\left| \int f_n - \int f \right| \leq \left(\int |f_n - f|^2 (1+x^2) \right)^{\frac{1}{2}} \left(\int (1+x^2)^{-1} \right)^{\frac{1}{2}} \to 0$.

$\Rightarrow \int f = \lim_{n\to\infty} \int f_n = 0$.

"$\supset$": let $f \in K$. We want to find $\{f_n\} \subset D(A)$ st $f_n \to f$ in L^2 and f_n is Cauchy sequent wrt $\|\sqrt{A}\cdot\|_{L^2}$.

Define $f_n(x) = f(x) \mathbb{1}_{[-n,n]}(x) - \frac{1}{2} \int_{-n}^{n} f(x)\,dx \; \mathbb{1}_{[-1,1]}(x)$

then one can check that $f_n \in D(A)$, $\sqrt{1+|x|^2}\, f_n$ is Cauchy sequence.