Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
Winter Semester 2019-2020
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Mathematical Quantum Mechanics

Midterm Exam

Nachname: ____________________ Vorname: ____________________

Geburtstag: ____________________ Matrikelnr.: ____________________

Studiengang: ____________________ Fachsemester: ____________________

Please place your identity and student ID cards on the table so that they are clearly visible. Switch off your mobile phone and all other electronic devices.

Please write your name and your solutions on the provided sheets. You can use your notes.

You have 100 minutes. Good luck!

Problems	1	2	3	4	5	$\sum$
Maximum points	2	2	2	2	2	10
Scored points						

Problem 1. (2 point) Let A be a bounded operator on a separable Hilbert space $\mathscr{H}$. Prove that the following statements are equivalent:

$$A \text{ is a compact operator} \quad \Longleftrightarrow \quad A^*A \text{ is a compact operator.}$$

Problem 2. (2 point) Let A be a self-adjoint operator on a separable Hilbert space $\mathscr{H}$ such that $(A+i)^{-1}$ is a compact operator.

(a) Prove that $\sigma_{\rm ess}(A) = \emptyset$.

(b) Prove that if A is also a bounded operator, then $\dim \mathscr{H} < \infty$.

Problem 3. (2 point) Let $t > 0$. Consider the heat kernel

$$K_t(x) = |4\pi t|^{-d/2} e^{-|x|^2/(4t)}, \quad x \in \mathbb{R}^d.$$

(a) Prove that $K_t * f \in H^m(\mathbb{R}^d)$ for all $m \in \mathbb{N}$ and for all $f \in L^2(\mathbb{R}^d)$.

(b) Prove that $\|K_t * f - f\|_{L^2(\mathbb{R}^d)} \le \sqrt{t}\|f\|_{H^1(\mathbb{R}^d)}$ for all $f \in H^1(\mathbb{R}^d)$.

Problem 4. (2 points) Consider the operator A on $L^2(0,1)$ defined by

$$(Af)(x) = (1+x^2)f(x) - \int_0^1 f(y)dy, \quad \forall f \in L^2(0,1).$$

Prove that A is a bounded self-adjoint operator and $\sigma_{\rm ess}(A) = [1,2]$.

Problem 5. (2 points) Consider the operator A on $L^2(\mathbb{R})$ defined by

$$(Af)(x) = (1+x^2)f(x), \quad D(A) = \left\{ f \in L^2(\mathbb{R}) : (1+x^2)f \in L^2(\mathbb{R}), \int_{\mathbb{R}} f = 0 \right\}.$$

We know that A is a (densely defined) closed operator (Homework 7.2).

a) What is the quadratic form domain of A? Justify your answer.

b) Let A_F be the Friedrichs extension of A. Prove that $\sigma(A_F) = [1,\infty)$.