Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
Winter Semester 2019/20
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Homework Sheet 9

(Due on 13.01.2020 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

9.1. Prove that $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ is dense in $H^1(\mathbb{R}^d)$ if $d \geq 3$, but not dense if $d = 1$.

9.2. Let $1 \leq p \leq q < r \leq \infty$. Prove that for any function $f \in L^q(\mathbb{R}^d)$ and any $\varepsilon > 0$ we can write

$$f = f_1 + f_2 \quad \text{with} \quad f_2 \in L^r(\mathbb{R}^d), \quad f_1 \in L^p(\mathbb{R}^d) \text{ and } \|f_1\|_{L^p} \leq \varepsilon.$$

9.3. Let μ be a positive measure on $\mathbb{R}^3$ with $0 < \mu(\mathbb{R}^3) < \infty$. Define

$$V(x) = (\mu * |\cdot|^{-1})(x) = \int \frac{d\mu(y)}{|x-y|}.$$

Consider the operator $A = -\Delta - V$ on $L^2(\mathbb{R}^3)$ with $D(A) = H^2(\mathbb{R}^3)$.

(i) Prove that A is self-adjoint and $\sigma_{\text{ess}}(A) = [0, \infty)$.
(ii) Prove that A has infinitely many negative eigenvalue.

9.4. Consider the operator $A : D(A) \to L^2(\mathbb{R}^3)$ defined by

$$A = -\Delta + \frac{1}{|x|} \quad \text{with} \quad D(A) = H^2(\mathbb{R}^3).$$

(i) Prove that $\sigma(A) = [0, \infty)$.
(ii) Prove that A has no eigenvalues.

9.5. Consider the operator $A = -\Delta + V$ on $L^2(\mathbb{R})$ with $0 \geq V \in C_c^\infty(\mathbb{R})$, $V \not\equiv 0$. Prove that A has at least one negative eigenvalue.