Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
Winter Semester 2019/20
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Homework Sheet 8

(Due on 16.12.2019 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

8.1. Let A be a self-adjoint compact operator on a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = \infty$. Prove that

$$\sigma_{\text{ess}}(A) = \{0\}.$$

8.2. Let K be a compact operator on a Hilbert space $\mathcal{H}$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of bounded self-adjoint operators on $\mathcal{H}$ such that

$$\lim_{n\to\infty} \|A_n u\| = 0, \quad \forall u \in \mathcal{H}.$$

Prove that $\|KA_n\| \to 0$.

8.3. Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that

$$\sigma_{\text{dis}}(A^2) \subset (\sigma_{\text{dis}}(A))^2 \quad \text{and} \quad \sigma_{\text{ess}}(A^2) = (\sigma_{\text{ess}}(A))^2.$$

Here by definition, $X^2 := \{\lambda^2 | \lambda \in X\}$.

8.4. Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a Hilbert space $(\mathcal{H}, \|.\|)$. Let $\lambda \in \sigma(A)$ and let $\{x_n\}_{n=1}^{\infty}$ be a Weyl sequence for (A, λ), namely

$$x_n \in D(A), \quad \|x_n\| = 1, \quad \|(A-\lambda)x_n\| \to 0.$$

Assume further that $x_n \rightharpoonup x$ weakly in $\mathcal{H}$ with $\|x\| < 1$. Prove that $\lambda \in \sigma_{\text{ess}}(A)$.

8.5. Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$ such that $A \geq 1$. Let Q_A be the quadratic form associated to A with the quadratic form domain $Q(A)$. Assume that there exists $u_0 \in Q(A)$, $\|u_0\| = 1$ such that

$$\|u_0\|_{Q_A} = \inf_{u \in Q(A), \|u\|=1} \|u\|_{Q_A}.$$

Prove that $u_0 \in D(A)$ and it is an eigenfunction of A.