Homework Sheet 7

(Due on 09.12.2019 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

7.1. Let $A : D(A) \to \mathcal{H}$ and $B : D(B) \to \mathcal{H}$ be self-adjoint operators on a Hilbert space $\mathcal{H}$. Prove that if $A \subset B$ then $A = B$.

7.2. Consider the operator $A : D(A) \to L^2(\mathbb{R})$ defined by

$$(Af)(x) = (1 + |x|^2)f(x), \quad \forall f \in D(A)$$

with

$$D(A) = \{f \in L^2(\mathbb{R}) : (1 + |x|^2)f \in L^2(\mathbb{R}), \int_{\mathbb{R}} f = 0\}.$$

(i) Prove that A is a closed symmetric operator.
(ii) Prove that A is not self-adjoint.

7.3. Consider the hydrogen operator $A = -\Delta - |x|^{-1}$ on $L^2(\mathbb{R}^3)$ with $D(A) = C_c^2(\mathbb{R}^3)$. Prove that $D(\overline{A}) = H^2(\mathbb{R}^3)$ and $\overline{A}$ is self-adjoint.

7.4. Let $A : D(A) \to \mathcal{H}$ and $B : D(B) \to \mathcal{H}$ be self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $A \geq 1$, $B \geq 1$. Let Q_A and Q_B be quadratic forms associated to A and B defined on the quadratic form domains $Q(A) = D(\sqrt{A})$ and $Q(B) = D(\sqrt{B})$, respectively. Assume that $Q_A = Q_B$, i.e., $Q(A) = Q(B)$ as sets and $\|\cdot\|_{Q_A} = \|\cdot\|_{Q_B}$. Prove that $A = B$.

Note: This "uniqueness part" in the Lax–Milgram theorem was not proved in the lecture.

7.5. Let $A : D(A) \to \mathcal{H}$ be a symmetric operator on a Hilbert space $\mathcal{H}$ such that $A \geq 1$. Let $B : D(A) \to \mathcal{H}$ be a symmetric operator. Assume that there exists a constant $\varepsilon > 0$ such that

$$|\langle x, Bx\rangle| \leq (1 - \varepsilon)\langle x, Ax\rangle + C_\varepsilon \|x\|^2, \quad \forall x \in D(A).$$

Prove that $A + B$ has a self-adjoint extension which has the same quadratic form domain of A.

Remark. This is an analogue of the Kato–Rellich theorem for quadratic forms.