Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
Winter Semester 2019/20
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Homework Sheet 6

(Due on 02.12.2019 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

In this homework we focus on some applications of Spectral Theorem. Here $i^2 = -1$.

6.1. Let A be a bounded self-adjoint operator on a Hilbert space. Prove that

$$\|A\| = \sup_{\|u\|=1} |\langle u, Au\rangle| \quad \text{and} \quad \|A^n\| = \|A\|^n,\ \forall n \in \mathbb{N}.$$

6.2. Let $A : D(A) \to \mathcal{H}$ be a (unbounded) self-adjoint operator on a separable Hilbert space $\mathcal{H}$.

i) Prove that $U = e^{iA}$ is a unitary operator on $\mathcal{H}$.
ii) Prove that $U = (A - i)(A + i)^{-1}$ is a unitary operator on $\mathcal{H}$.

6.3. Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a separable Hilbert space. For any $n \in \mathbb{N}$ consider

$$A_n = n(A + in)^{-1} + i, \quad n \in \mathbb{N}.$$

i) Prove that $\|A_n u\| \to 0$ as $n \to \infty$ for any $u \in \mathcal{H}$.
ii) Prove that $\|A_n\| \to 0$ as $n \to \infty$ if and only if A is bounded.

6.4. Let $\{A_n\}_{n=1}^{\infty}$ and A be self-adjoint operators on a separable Hilbert space $\mathcal{H}$ with the same domain $D(A_n) = D(A)$ such that

$$\lim_{n\to\infty} \|A_n\varphi - A\varphi\| = 0, \quad \forall \varphi \in D(A).$$

Prove that

$$\lim_{n\to\infty} \|(A_n + i)^{-1}u - (A + i)^{-1}u\| = 0, \quad \forall u \in \mathcal{H}.$$

6.5. Let (Ω, μ) be a measure space. Let M_f be the multiplication operator on $L^2(\Omega)$ of a measurable function $f : \Omega \to \mathbb{C}$ with the natural domain

$$D(M_f) = \{g \in L^2(\Omega) : fg \in L^2(\Omega)\}.$$

Prove that $\lambda \in \mathbb{C}$ is an eigenvalue of M_f if and only if the set $E = \{x \in \Omega : f(x) = \lambda\}$ has positive measure.