Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
Winter Semester 2019/20
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Homework Sheet 5

(Due on 25.11.2019 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

5.1. Let A be a bounded linear operator on a Hilbert space. Prove that the following statements are equivalent

i) A is a compact operator;
ii) $x_n \rightharpoonup x$ weakly implies $Ax_n \to Ax$ strongly.

5.2. Let $\{u_n\}_{n=1}^\infty, \{v_n\}_{n=1}^\infty$ be orthonormal families in a Hilbert space. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of complex numbers. Consider the operator

$$A = \sum_{n=1}^\infty \lambda_n |v_n\rangle\langle u_n|.$$

i) Prove that if $\{\lambda_n\}$ is bounded, then A is bounded and $\|A\| = \sup_{n\geq 1} |\lambda_n|$.
ii) Prove that if $\lambda_n \to 0$ as $n \to \infty$, then A is a compact operator.

5.3. Let A be a bounded operator on a Hilbert space $\mathcal{H}$. Let V be a closed subspace of $\mathcal{H}$ such that $A : V \to V$. Prove that $A^* : V^\perp \to V^\perp$.

5.4. Let A be a bounded operator on a Hilbert space $\mathcal{H}$ such that

$$\|Au\| \geq \|u\| \quad \text{and} \quad \|A^*u\| \geq \|u\|, \quad \forall u \in \mathcal{H}.$$

Prove that A^{-1} is a bounded operator.

5.5. Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$.

i) Assume that there exists a vector $u_0 \in \mathcal{H}$ such that $\|u_0\| \leq 1$ and

$$\langle u_0, Au_0\rangle = \inf_{\|u\|\leq 1} \langle u, Au\rangle =: E.$$

Prove that $Au_0 = Eu_0$.

ii) Deduce that if $\langle u, Au\rangle = 0$ for all $u \in \mathcal{H}$, then $A \equiv 0$.

Hint: for (i) you can use $\langle u_0, Au_0\rangle \leq \langle u_\varepsilon, Au_\varepsilon\rangle$ with $u_\varepsilon = (u_0 + \varepsilon\varphi)/\|u_0 + \varepsilon\varphi\|$ for $|\varepsilon|$ small.

5.6. Let (Ω, μ) be a measure space. Let M_a be the multiplication operator on $L^2(\Omega)$ associated with a function $a \in L^\infty(\Omega)$. Prove that the spectrum $\sigma(M_a)$ is equal to the essential range of a.