Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
Winter Semester 2019/20
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Homework Sheet 4

(Due on 18.11.2019 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

4.1. Let H be a Hilbert space.

(i) Prove the parallelogram law

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2), \quad \forall f, g \in H.$$

(ii) Prove that H is *uniformly convex*, namely if $\|f\| \le 1, \|g\| \le 1$ and $\|f+g\| \ge 2-\delta$ for some $\delta > 0$, then

$$\|f-g\| \le \varepsilon_\delta.$$

Here $\varepsilon_\delta > 0$ is a constant depending only on δ such that $\lim_{\delta\to 0} \varepsilon_\delta = 0$.

Remark: The above result tells us that $L^2(\Omega)$ is uniformly convex. More generally, $L^p(\Omega)$ is uniformly convex for all $1 < p < \infty$, by Clarkson's theorem.

4.2. Let (Ω, μ) be a measure space. Assume that $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ with $1 < p < \infty$.

(i) Prove that

$$\liminf_{n\to\infty} \|f_n\|_{L^p} \ge \|f\|_{L^p}.$$

(ii) Assume further that $\|f_n\|_{L^p} \to \|f\|_{L^p}$. Prove that $f_n \to f$ strongly in $L^p(\Omega)$.

Hint: You can use Clarkson's theorem (without proof).

4.3. Let H be a Hilbert space. Let $f_n \rightharpoonup 0$ weakly in H.

(i) Prove that there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ such that

$$|\langle f_{n_k}, f_{n_\ell}\rangle| \le 2^{-k}, \quad \forall \ell > k.$$

(ii) Prove that the Cesàro (arithmetic) mean of $\{f_{n_k}\}_{k=1}^\infty$ converges strongly to 0, namely

$$\lim_{K\to\infty} \left\| \frac{1}{K} \sum_{k=1}^K f_{n_k} \right\| = 0$$

Remark: More generally the Banach–Saks theorem states that: if $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ with $1 < p < \infty$, then there exists a subsequence whose Cesàro mean converges strongly to f (this property also holds for uniformly convex Banach spaces, by Kakutani's theorem).

4.4. Let (Ω, μ) be a measure space. Assume that $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ for some $1 < p < \infty$ and $f_n(x) \to g(x)$ for a.e. $x \in \Omega$. Prove that $f = g$.

Hint: You can use the Banach–Saks theorem (without proof).

4.5. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a measurable, bounded function with compact support. Prove that for all $g \in C_c^\infty(\mathbb{R}^d)$, the convolution $f * g$ belongs to $C_c^\infty(\mathbb{R}^d)$.

4.6. For any $t > 0$, consider the heat operator $K_t = e^{t\Delta}$ on $L^2(\mathbb{R}^d)$ defined by

$$\widehat{(K_t f)}(k) = e^{-t|2\pi k|^2} \widehat{f}(k), \quad \forall f \in L^2(\mathbb{R}^d).$$

Prove that

$$(K_t f)(x) = \int_{\mathbb{R}^d} K_t(x-y) f(y) dy, \quad \text{with } K_t(x) = \frac{e^{-|x|^2/(4t)}}{|4\pi t|^{d/2}}.$$

Hint: You can use the fact that the Fourier transform of $e^{-\pi|x|^2}$ is $e^{-\pi|k|^2}$.