Mathematisches Institut der LMU
Prof. P. T. Nam, Prof. A. Scrinzi
S. Balkan, D.T. Nguyen

Mathematical Quantum Mechanics
Winter Semester 2019/20
4. 11. 2019

Homework Sheet 3

(Due on 11.11.2019 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

3.1. Let (Ω, Σ, μ) be a measure space. Let $A = \cap_{n=1}^{\infty} A_n$ with $A_n \in \Sigma$ for all $n = 1, 2, ...$

(i) Prove that $A \in \Sigma$.

(ii) Prove that if $A_1 \supset A_2 \supset ... \supset A_n \supset ...$ and $\mu(A_1) < \infty$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

(iii) Can we relax the condition $\mu(A_1) < \infty$ in (ii)?

3.2. Let (Ω, Σ, μ) be a measure space and let $f : \Omega \to [0, \infty]$. Prove that the following statements are equivalent

(i) f is measurable, i.e. $\{x \in \Omega : f(x) > t\} \in \Sigma$ for all $t \in \mathbb{R}$.

(ii) $\{x \in \Omega : f(x) \geq t\} \in \Sigma$ for all $t \in \mathbb{R}$.

Hint: You can write $\{x : f(x) \geq t\} = \cap_{n=1}^{\infty} \{x : f(x) > t - n^{-1}\}$.

3.3. Let (Ω, μ) be a measure space. Let $f_n : \Omega \to [0, \infty]$ be measurable functions.

(i) Define $F_n(x) = \sup_{m\geq n} f_m(x)$. Prove that F_n is measurable for all $n = 1, 2, ...$

(ii) Prove that if $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then $\{x : f(x) \geq t\} = \cap_{n=1}^{\infty} \{x : F_n(x) \geq t\}$ and hence f is measurable.

Note: This exercise shows that the pointwise limit of measurable functions are measurable.

3.4. Let (Ω, μ) be a measure space and let $f : \Omega \to \mathbb{C}$ be measurable. Prove that

$$\int_\Omega |f(x)|^p d\mu(x) = p \int_0^\infty t^{p-1} \mu(\{x \in \Omega : |f(x)| > t\}) dt, \quad \forall 1 \leq p < \infty.$$

3.5. Let (Ω, μ) be a measure space. Let $\{f_n\}_{n=1}^{\infty}$ be measurable functions and f be an integrable function such that $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x. Prove that

$$\lim_{n\to\infty} \int_\Omega \big||f_n(x)| - |f(x)| - |f_n(x) - f(x)|\big| d\mu(x) = 0.$$

Note: This is the Brezis–Lieb lemma with $p = 1$, but we do not assume $\int_\Omega |f_n| \leq C$.

3.6. Given subsets A, B of $\mathbb{R}$ such that

(i) A is bounded and not Lebesgue-measurable, and

(ii) B is Lebesgue measurable and $\mu(B) = 0$.

Prove that $A \times B$ is Lebesgue-measurable in $\mathbb{R}^2$.

Note: The 1D projection of a 2D Lebesgue-measurable set is not necessarily measurable.