Mathematisches Institut der LMU
Prof. P. T. Nam, Prof. A. Scrinzi
D.T. Nguyen, S. Balkan

Mathematical Quantum Mechanics
Winter Semester 2019/20
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Homework Sheet 2

(Due on 04.11.2019 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

2.1. Let $\{x_n\}$ be an orthonormal family in a separable Hilbert space H. Prove that $x_n \rightharpoonup 0$ weakly, i.e.

$$\lim_{n\to\infty} \langle x_n, y\rangle = 0, \quad \forall y \in H.$$

2.2. Let A be a bounded operator on a separable Hilbert space H. Prove that if $x_n \rightharpoonup x$ weakly in H, then $Ax_n \rightharpoonup Ax$ weakly in H.

2.3. Let $H = L^2(\Omega)$ with (Ω, μ) a measure space. Let $f : \Omega \to \mathbb{C}$ be a measurable function. Define the multiplication operator M_f as an (unbounded) operator on H as follows

$$D(M_f) = \{\varphi \in L^2(\Omega) : f\varphi \in L^2(\Omega)\}, \quad (M_f(\varphi))(x) = f(x)\varphi(x).$$

(i) Prove that $D(M_f)$ is dense in H.
(ii) Prove that $(M_f)^* = M_{\overline{f}}$ with the same domain $D((M_f)^*) = D(M_f)$.

2.4. Let (Ω, μ) be a measurable space and let $1 \leq p < q < r \leq \infty$.

(i) Prove that

$$\big(L^p(\Omega) \cap L^r(\Omega)\big) \subset L^q(\Omega).$$

(ii) Prove that if we assume further $\mu(\Omega) < \infty$, then

$$L^r(\Omega) \subset L^q(\Omega) \subset L^p(\Omega).$$

Find a counter-example that the conclusion is wrong if $\mu(\Omega) = +\infty$.

2.5. Construct an example for $f \in L^2(\mathbb{R})$ (with the usual Lebesgue measure) but $f \notin L^p(\mathbb{R})$ for any $2 \neq p \in [1, \infty]$.