Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
Winter Semester 2019/20
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Homework Sheet 12

(Due on 03.02.2020 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

12.1. Let A be a self-adjoint operators on a separable Hilbert space $\mathcal{H}$. Let $\{K_n\}$ be a sequence of compact operators such that $\lim_{n\to\infty} \|K_n u - u\| = 0$ for all $u \in \mathcal{H}$. Prove that for any $u_0 \in H_p$ (the subspace spanned by eigenfunctions of A) we have

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|(1-K_n)e^{-itA}u_0\| = 0.$$

Remark. This is a special case of RAGE theorem.

12.2. Let A be a self-adjoint operators on a separable Hilbert space $\mathcal{H}$. Prove that for any $u_0 \in \mathrm{Ker}(A)^\perp$ we have

$$\frac{1}{T}\int_0^T e^{-itA}u_0 \mathrm{d}t \rightharpoonup 0$$

weakly in $\mathcal{H}$ as $T \to \infty$.

12.3. Let A and B be self-adjoint operators on a separable Hilbert space $\mathcal{H}$. Assume that the wave operator $\Omega := \lim_{t\to\infty} e^{itA}e^{-itB}$ exists, namely for any $u_0 \in \mathcal{H}$,

$$\lim_{t\to\infty} e^{itA}e^{-itB}u_0 = \Omega u_0$$

strongly in $\mathcal{H}$. Prove that if A has no eigenvalue then B also has no eigenvalue.

12.4. Let $V \in L^2 + L^p(\mathbb{R}^3)$ with $2 \le p < 3$. Prove that for any $u_0 \in L^1 \cap L^2(\mathbb{R}^3)$ we have

$$\int_{-\infty}^{-1} \|Ve^{-it\Delta}u_0\|_{L^2(\mathbb{R}^3)}\mathrm{d}t + \int_1^{\infty} \|Ve^{-it\Delta}u_0\|_{L^2(\mathbb{R}^3)}\mathrm{d}t < \infty$$

Remark. This implies the existence of wave operators $\lim_{t\to\infty} e^{it(-\Delta+V)}e^{it\Delta}$, $\lim_{t\to-\infty} e^{it(-\Delta+V)}e^{it\Delta}$.