Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
Winter Semester 2019/20
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Homework Sheet 11

(Due on 27.01.2020 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

11.1. Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a separable Hilbert space $(\mathcal{H}, \|\cdot\|)$. Consider the Schrödinger equation

$$\begin{aligned} i\partial_t \psi(t) &= A\psi(t), \quad t \in \mathbb{R} \\ \psi(0) &= \psi_0 \in D(A) \end{aligned}$$

Prove that for any $t \in \mathbb{R}$ we have

$$\|A\psi(t)\| = \|A\psi_0\| \quad \text{and} \quad \langle \psi(t), A\psi(t)\rangle = \langle \psi_0, A\psi_0\rangle.$$

11.2. Let $1 \leq d \leq 3$. Let $V : \mathbb{R}^d \to \mathbb{R}$ such that $V \in L^2 + L^p(\mathbb{R}^d)$ with $2 \leq p \leq \infty$. Consider the Schrödinger equation

$$\begin{aligned} i\partial_t \psi(t,x) &= (-\Delta_x + V(x))\,\psi(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d \\ \psi(0,x) &= \psi_0(x). \end{aligned}$$

(i) Prove that if $\psi_0 \in H^2(\mathbb{R}^d)$ then $\{\psi(t)\}_{t\in\mathbb{R}}$ is bounded in $H^2(\mathbb{R}^d)$.
(ii) Prove that if $\psi_0 \in H^1(\mathbb{R}^d)$ then $\{\psi(t)\}_{t\in\mathbb{R}}$ is bounded in $H^1(\mathbb{R}^d)$.

11.3. Let A be a self-adjoint operator on $L^2(\mathbb{R}^d)$ with orthonormal eigenfunctions $\{u_n\}_{n=1}^{\infty}$, namely $Au_n = \lambda_n u_n$ with $\{\lambda_n\} \subset \mathbb{R}$. Consider the Schrödinger equation

$$\begin{aligned} i\partial_t \psi(t,x) &= A\psi(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d \\ \psi(0,x) &= \psi_0(x). \end{aligned}$$

(i) Compute $\psi(t)$ with initial state

$$\psi_0 = \sum_{n=1}^{\infty} \alpha_n u_n \quad \text{where} \quad \alpha_n \in \mathbb{C} \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_n|^2 = 1.$$

(ii) Prove that

$$\lim_{R\to\infty} \inf_{t\in\mathbb{R}} \int_{|x|\leq R} |\psi(t,x)|^2 \,\mathrm{d}x = 1.$$

11.4. Consider the free Schrödinger equation

$$u(t) = e^{it\Delta} u_0, \quad u_0 \in L^2(\mathbb{R}^d).$$

Prove that $u(t) \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^d)$ as $|t| \to \infty$.