Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
Winter Semester 2019/20
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Homework Sheet 10

(Due on 20.01.2020 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

10.1. Let $V \in C_c^\infty(\mathbb{R}^d)$ with $d \geq 3$ such that $V \geq 0$ and $V \not\equiv 0$. For any $\lambda > 0$, consider the Schrödinger operator $A_\lambda = -\Delta - \lambda V(x)$ on $L^2(\mathbb{R}^d)$ (which is a self-adjoint operator with $D(A) = H^2(\mathbb{R}^d)$). Denote by N_λ the number of negative eigenvalues of A_λ.

(i) Prove that $N_\lambda \to 0$ as $\lambda \to 0$.
(ii) Prove that $N_\lambda \to \infty$ as $\lambda \to \infty$.

10.2. For any $\lambda > 0$, consider the Schrödinger operator

$$A_\lambda = -\Delta - \mathbb{1}(|x| \leq 1) + \lambda$$

on $L^2(\mathbb{R}^2)$ (which is a self-adjoint operator with $D(A) = H^2(\mathbb{R}^2)$). Let N_λ be the number of negative eigenvalues of A_λ. Prove that for any $\varepsilon > 0$, there exists a constant C_ε depending only on ε such that

$$N_\lambda \leq C_\varepsilon \lambda^{-\varepsilon}, \quad \forall \lambda > 0.$$

Hint. You can mimic the proof of CLR bound.

10.3. Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Assume that A is bounded from below. Prove that the following statements are equivalent.

(i) A has compact resolvent, namely $(A - z)^{-1}$ is a compact operator for all $z \in \rho(A)$ (the resolvent set of A).
(ii) The min-max values $\{\mu_n\}_{n=1}^\infty$ of A satisfy $\lim_{n\to\infty} \mu_n = +\infty$.
(iii) A has eigenvalues $\{\lambda_n\}_{n=1}^\infty$ with eigenvectors $\{x_n\}_{n=1}^\infty$, i.e. $Ax_n = \lambda_n x_n$, such that $\lim_{n\to\infty} \lambda_n = +\infty$ and $\{x_n\}_{n=1}^\infty$ is an orthonormal basis for $\mathcal{H}$.

10.4. Let $V : \mathbb{R}^d \to \mathbb{R}$ such that $V \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ and $\lim_{|x|\to\infty} V(x) = +\infty$. Recall that by Friedrichs' method, we can define

$$A = -\Delta + V(x)$$

as a self-adjoint operator on $L^2(\mathbb{R}^d)$ with domain $D(A) \supset C_c^\infty(\mathbb{R}^d)$. Prove that A has compact resolvent.