Mathematisches Institut der LMU
Prof. P. T. Nam, Prof. A. Scrinzi
D.T. Nguyen, S. Balkan

Mathematical Quantum Mechanics
Winter Semester 2019/20
21. 10. 2019

Homework Sheet 1

(Due on 28.10.2019 by 14:15 in the box "Mathematical Quantum Mechanics", no 45)

1.1. Let A be a bounded operator on a separable Hilbert space. Prove that

(i) $\|A\| = \|A^*\|$.
(ii) $\|A^*A\| = \|A\|^2$.

Hint: You may use the formula $\|x\| = \sup_{\|y\|\le 1} |\langle x, y\rangle|$.

1.2. Let H be a separable Hilbert space and let $A : D(A) \to H$ be a (densely defined) unbounded operator. Prove that the followings are equivalent:

(i) A is symmetric, i.e. $\langle x, Ay\rangle = \langle Ax, y\rangle$ for all $x, y \in D(A)$;
(ii) $\langle x, Ax\rangle \in \mathbb{R}$ for all $x \in D(A)$.

1.3. (i) Prove Heisenberg's uncertainty principle (for all dimensions $d \ge 1$)

$$\left(\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx\right)\left(\int_{\mathbb{R}^d} |x|^2 |u(x)|^2 \mathrm{d}x\right) \ge \frac{d^2}{4}, \quad \forall u \in C_\mathrm{c}^1(\mathbb{R}^d), \int_{\mathbb{R}^d} |u(x)|^2 \mathrm{d}x = 1.$$

Hint: You can use the commutator relation $\nabla \cdot x - x \cdot \nabla = d$.

(ii) By Heisenberg's uncertainty principle we can bound the energy of hydrogen atom (with a wave function $u : \mathbb{R}^3 \to \mathbb{C}$) from below by

$$\mathcal{E}(u) = \frac{9}{4}\left(\int_{\mathbb{R}^3} |x|^2 |u(x)|^2 \mathrm{d}x\right)^{-1} - \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} \mathrm{d}x.$$

Determine $\inf\{\mathcal{E}(u) : u \in C_\mathrm{c}^1(\mathbb{R}^3), \int_{\mathbb{R}^3} |u(x)|^2 \mathrm{d}x = 1\}$.

1.4. Here we examine the optimality of Hardy's inequality.

(i) Prove that for any constant $a > 2$, there exists no constant $C > 0$ such that

$$\int_{\mathbb{R}^3} |\nabla u(x)|^2 \mathrm{d}x \ge C \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^a} \mathrm{d}x, \quad \forall u \in C_c^1(\mathbb{R}^3).$$

(ii) Prove that for any constant $C > 1/4$, there exists a function $f \in C_c^1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} |\nabla f(x)|^2 \mathrm{d}x < C \int_{\mathbb{R}^3} \frac{|f(x)|^2}{|x|^2} \mathrm{d}x.$$