Tutorial 9 for 14/12/2018

Ex1: Let A be a self-adjoint operator and let $U(t) = e^{-itA}$, $t \in \mathbb{R}$. Prove that

i) If A is bounded then $\lim_{t\to 0} \| U(t) - 1 \| = 0$

ii) If $\lim_{t\to 0} \frac{U(t)\psi - \psi}{t}$ exists then $\psi \in D(A)$

Hint: Define $D(B) = \{ \psi \in H : \lim_{t\to 0} \frac{U(t)\psi - \psi}{t} \text{ exists} \}$

$iB\psi = \lim_{t\to 0} \frac{U(t)\psi - \psi}{t}$ then B is symmetric and $A \subset B$.

Ex2: Let $A : D(A) \to H$ be a self-adjoint operator, $A \geq 0$

$B : D(A) \to H$ be a symmetric operator

Prove that $B(A+1)^{-1}$ is compact

$\Leftrightarrow$ B is A-relatively compact, i.e. $B(A+i)^{-1}$ is compact.

Ex3: Let $A : D(A) \to H$ and $B : D(B) \to H$ be two self-adjoint operator, and $A \geq B \geq 0$. Prove that

i) $C^*AC \geq C^*BC$ for all operator $C : D(C) \to D(A) \cap D(B)$

ii) For any $\lambda > 0$, $A + \lambda$ and $B + \lambda$ are invertible and

$$(A+\lambda)^{-1} \leq (B+\lambda)^{-1} \leq \lambda^{-1}$$

iii) Neither $A^2 \geq B^2$ nor $AB \geq 0$.

Note: we have $A^s \geq B^s \geq 0$ $\forall s \in (0,1)$.