Tutorial 5 for 16 November 2018

Theorem: Let $x_n \to x$ weakly in Hilbert space $\mathcal{H}$. Then x_n is bounded uniformly in $\mathcal{H}$.

Proof: $\langle x_n, y\rangle \to \langle x, y\rangle\ \forall y \in \mathcal{H} \Rightarrow \langle x_n, y\rangle$ is bdd uniformly.

Let $T_n : \mathcal{H} \to \mathbb{C}$, $y \mapsto \langle x_n, y\rangle$ is linear, bdd, i.e. $\sup_n |T_n(y)| < \infty$ $\forall y \in \mathcal{H}$.

By Uniformly Boundedness principle / Banach-Steinhaus theorem

$$\infty > \sup_n \|T_n\| = \sup_n \sup_{\|y\|\le 1} |T_n(y)| = \sup_n \|x_n\|.$$

Proof of UBP (in Hilbert space).

Lemma: For any $x \in \mathcal{H}$ and $r>0$ we have

$$\sup_{x' \in B(x,r)} \|Tx'\| \geq r\|T\| \qquad \square$$

Now, assume that $\sup_n \|T_n\| = \infty$ and choose sequence (T_n): $\|T_n\| \geq 4^n$. We use the Lemma to choose inductively $x_n \in \mathcal{H}$ st $\|x_n - x_{n-1}\| \leq 3^{-n}$ and $\|T_n x_n\| \geq \frac{2}{3} 3^{-n} \|T_n\|$.

It follows that $\{x_n\}$ is Cauchy sequence since

$$\forall m>n: \|x_m - x_n\| \leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \ldots + \|x_{n+1} - x_n\|$$
$$\leq 3^{-m} + 3^{-m+1} + \ldots + 3^{-n-1} = \frac{1}{2} 3^{-n}(1 - 3^{-m+n}) \leq \frac{1}{2} 3^{-n}.$$

Then $x_n \to x$ as $n \to \infty$. Taking the limit $m \to \infty$ we have

$$\|x - x_n\| \leq \frac{1}{2} 3^{-n}.$$

Hence

$$\|T_n x\| \geq \|T_n x_n\| - \|T_n(x_n - x)\| \geq \frac{2}{3} 3^{-n}\|T_n\| - \frac{1}{2} 3^{-n}\|T_n\|$$
$$\geq \frac{1}{6} 3^{-n}\|T_n\| \geq \frac{1}{6}\left(\frac{4}{3}\right)^n \longrightarrow \infty \text{ as } n \to \infty$$

This contradicts to $\sup_n |T_n(x)| < \infty\ \forall x \in \mathcal{H}$.

Ex 1: Let A be a symmetric operator on a separable Hilbert space $\mathcal{H}$. Prove that if $\langle u, Au\rangle = 0$ $\forall u \in H$ then $A \equiv 0$.

Ex 2: Let P $(P \neq 0, P \neq id)$ be a projection on $\mathcal{H}$. Show that
$$\sigma(P) = \{0, 1\}.$$

Ex 3: Let $\{u_n\}, \{v_n\}$ be ONF for $\mathcal{H}$, and let
$$A = \sum_{n=1}^{\infty} \lambda_n |u_n\rangle\langle u_n| \text{ with } |\lambda_n| \to 0 \text{ as } n \to \infty.$$
Prove that A is compact operator.

Ex 4: Let T be a distribution. Assume that there exists $C > 0$ st $|T(\varphi)| \leq C\|\varphi\|_{L^2(\mathbb{R}^d)}$ $\forall \varphi \in C_c^\infty(\mathbb{R}^d)$.
Prove that there exist $f \in L^2(\mathbb{R}^d)$ st $T = T_f$.