Tutorial 2 for 26/10/2018

Ex 1: Let $\mu(\Omega) < \infty$. Prove that if $1 \leq p < q \leq \infty$ then
$$L^q(\Omega) \subset L^p(\Omega).$$
Find a counter-example that the conclusion is wrong if $\mu(\Omega) = \infty$.

Definition: $L^p_{loc}(\Omega) = \{f \in L^p(\Omega) : \int_K |f|^p < \infty \ \forall \text{ compact } K \subset \Omega\}$

THM: $L^\infty_{loc}(\Omega) \subset L^q_{loc}(\Omega) \subset L^p_{loc}(\Omega) \subset L^1_{loc}(\Omega)$
for $1 \leq p < q \leq \infty$.

Ex 2: Let $f \in L^1(\mathbb{R}^d)$. Prove that
$$\lim_{|h| \to 0} \int |f(x+h) - f(x)| \, dx = 0.$$

Ex 3: Let $f \in L^1(\mathbb{R}^d)$. Prove that $\hat{f} \in L^\infty(\mathbb{R}^d)$ and $\hat{f} \in C(\mathbb{R}^d)$.

Ex 4: Let $x, y \in \mathcal{H}$ (Hilbert space). Prove the C-S ineq
$$|\langle x, y\rangle| \leq \|x\| \, \|y\|.$$

Ex 5: (Bessel ineq) Let $\{e_n\}$ be ONF in $\mathcal{H}$. Prove that for any $x \in \mathcal{H}$ we have
$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2$$

~~Ex 6 Prove that weak limit is unique.~~