

Tutorial 1 for 19/10/2018

Def: A measure μ is a function Σ (sigma-algebra) $\rightarrow [0, \infty]$ such that:

- i) $\mu(A) \geq 0$ for all $A \in \Sigma$
- ii) If $(A_n)_{n=1}^{\infty} \subset \Sigma$ such that $A_n \cap A_m = \emptyset$ then
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A_i)$$

Ex: a) Prove that if $A \subset B \in \Sigma$ then $\mu(A) \leq \mu(B)$

b) Prove that for ~~any~~ ~~any~~ $A_n \in \Sigma$ (not necessarily disjoint) we have $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

Hint: Write $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$.

THM (Approximation by Continuous Function). C_c is dense in L^1 .

More precisely, for any $f \in L^1(\mathbb{R}^n)$ there exists a sequence of continuous functions with compact support $\{f_n\}$ such that
$$\lim_{n \rightarrow \infty} \int |f_n - f| dx = 0.$$

Proof. Step 1: L_c^1 (space of integrable function with compact support) is dense in L^1 since
$$\lim_{k \rightarrow \infty} \int |f \chi_{B(0,k)} - f| dx = 0 \text{ by Dominated CV}$$

Step 2. $f = (\operatorname{Re} f)_+ - (\operatorname{Re} f)_- + i((\operatorname{Im} f)_+ - (\operatorname{Im} f)_-)$

It follows that if we successfully approximate $(\operatorname{Re} f)_+$, $(\operatorname{Re} f)_-$, $(\operatorname{Im} f)_+$, $(\operatorname{Im} f)_-$ then we can approximate f also. Thus we can assume $f \geq 0$ and f vanishes outside a bounded set.

Step 3. There exists a sequence of measurable simple functions $0 \leq s_1 \leq s_2 \leq \dots \leq f$ such that $\lim_{k \rightarrow \infty} s_k = f$. By Monotone CV
$$\lim_{k \rightarrow \infty} \int |s_k - f| dx = 0$$

We approximate a measurable simple function which is nonnegative and vanishes outside a bounded set. (cube)

Step 4. $\phi_k = \sum_{k=1}^m \alpha_k \chi_{A_k}$ where $\alpha_k \in [0, \infty)$ and $A_k \in \mathcal{L}$ (measurable sets). We only need to approximate χ_A by a function in $\mathcal{C}$, for every $A \in \mathcal{L}$.

Step 5: ~~For~~ $A \in \mathcal{L}$, given any ε , (we can assume G is bd)

Outer regularity: $\exists$ an open set $G \supset A$ st $\mu(G \setminus A) < \frac{\varepsilon}{2}$

Inner regularity: $\exists$ close/compact set $K \subset A$ st $\mu(A \setminus K) < \frac{\varepsilon}{2}$

$$\Rightarrow \mu(G \setminus K) < \varepsilon.$$

Define function $d(x, G^c) =: g(x)$ (Urysohn function)

$$\begin{cases} = 0 & \forall x \in G^c \\ = 1 & \forall x \in K \\ \in [0, 1] & \forall x \in \mathbb{R}^n. \end{cases}$$

g is continuous since $d(x, A)$ is continuous. Observe $g(x) - \chi_A(x) = 0$ if $x \in K$ or $x \in G^c$.

$$\text{Thus } \int |g - \chi_A| dx = \int_{G \setminus K} |g - \chi_A| dx \leq 2\mu(G \setminus K) < 2\varepsilon.$$

Ex: $d(x, A)$ is continuous function of x .

THM (Hölder's inequality): let $1 \leq p \leq \infty$, $\frac{1}{q} + \frac{1}{p} = 1$.

Assume that $f \in L^p$ and $g \in L^q$. Then $fg \in L^1$ and

$$\int |fg| \leq \|f\|_{L^p} \|g\|_{L^q}$$

Proof: If $p=1$ or $p=\infty$: it is obvious.

If $0 < p < \infty$: using Young's inequality

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q \quad \forall a \geq 0, b \geq 0.$$

Choosing $a = |f| / \|f\|_{L^p}$, $b = |g| / \|g\|_{L^q}$.

Ex: Prove Young's inequality

Hint: $t \mapsto \log t$ is concave on $(0, \infty)$

Ex. If $\|f\|_p = 0$ then $f = 0$ a.e.

Hint: $A = \{x: f(x) > 0\} = \bigcup_{k=1}^{\infty} \{x: f(x) > \frac{1}{k}\} = \bigcup_{k=1}^{\infty} A_k$

Show that $\mu(A_k) = 0$ and $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k) = 0$.

Ex: Prove by induction the following extension of Hölder's inequality

$$\|f_1 f_2 \dots f_k\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \dots \|f_k\|_{p_k}$$

where $f_i \in L^{p_i}$, $\forall 1 \leq i \leq k$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_k} \leq 1$

In particular, if $k=2$, $f_1 = f^\alpha$, $f_2 = f^{1-\alpha}$, $0 < \alpha < 1$
Then the following interpolation inequality holds

$$\|f\|_p \leq \|f\|_r^\alpha \|f\|_s^{1-\alpha} \text{ where } \frac{1}{p} = \frac{\alpha}{r} + \frac{1-\alpha}{s}$$

THM: $L^r \cap L^s \subset L^p$ for $1 \leq r \leq p \leq s \leq \infty$.

Ex (Minkowski inequality) For $f, g \in L^p$ we have

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \quad \forall 1 \leq p \leq \infty$$

Hint: Write $|f+g|^p = |f+g|^{p-1} |f+g|$ and use Hölder's inequality.

Ex (Brezis-Lieb lemma for $0 < p < 1$) let f_n be a sequence of ~~integrable~~ ^{in L^p ($0 < p < 1$)} function and $f_n \rightarrow f$ a.e. Prove that

$$\int |f_n|^p - |f|^p - |f_n - f|^p dx \xrightarrow{n \rightarrow \infty} 0$$

Hint: ~~Use~~ $|a+b|^p \leq |a|^p + |b|^p \quad \forall 0 < p < 1$.

Ex: Find an ~~example~~ that $f_n \rightarrow f$ a.e. but $\int f_n \not\rightarrow \int f$

Hint: $f_n(x) = n^2 \mathbb{1}_{(0, n^{-1})}$.
Ex: (~~Extension~~ Fatou lemma): ~~let~~ f_n be a sequence of

integrable function, $f_n \xrightarrow{n \rightarrow \infty} f$ a.e., and $f_n \geq g$ for an integrable function g . Then

$$\liminf_{n \rightarrow \infty} \int_{\Omega} f_n(x) d\mu(x) \geq \int_{\Omega} f(x) d\mu(x).$$

Ex (Reverse Fatou lemma). Let f_n be a sequence of integrable function, $f_n \xrightarrow{n \rightarrow \infty} f$ a.e., $f_n \leq g$ for an integrable function g . Then

$$\limsup_{n \rightarrow \infty} \int_{\Omega} f_n(x) d\mu(x) \leq \int_{\Omega} f(x) d\mu(x)$$

THM: (Completeness of L^p) Let f_n be a Cauchy sequence in L^p . Prove that there exists $f \in L^p$ such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0.$$

Proof: Step 1: If f_n is Cauchy sequence in L^p and the subsequence f_{n_k} of f_n converges to f , then $f_n \xrightarrow[n \rightarrow \infty]{L^p} f$

Step 2: $\forall \epsilon > 0, \exists N(\epsilon)$ st $\|f_k - f_j\|_p \leq \epsilon \quad \forall j, k \geq N(\epsilon)$.

Choose the sequence $\{n_k\}$ as follow

$$n_1 \geq N(2^{-1}), \quad n_k \geq N(2^{-k}) \text{ and } n_k > n_{k-1}.$$

It follow that $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}$ since $n_{k+1} > n_k \geq N(2^{-k})$

Step 3: Define function $F = |f_1| + \sum_{j=1}^{\infty} |f_{j+1} - f_j|$.

Minkowski's ineq implies $\|F\|_p \leq \|f_1\|_p + \sum_{j=1}^{\infty} \|f_{j+1} - f_j\|_p$

$$\Rightarrow \int F^p < \infty \quad \leq \|f_1\|_p + \sum_{j=1}^{\infty} 2^{-j} < \infty.$$

$\Rightarrow F < \infty$ a.e. This means $\exists$ set $A \subset \Omega$ st $\mu(A) = 0$ and $F(x) < \infty \quad \forall x \in \Omega \setminus A$. Thus, if

$x \in \Omega \setminus A$ we have $|f_1(x)| + \sum_{j=1}^{\infty} |f_{j+1}(x) - f_j(x)|$ converges

$$\Rightarrow \lim_{k \rightarrow \infty} \left\{ f_1(x) + \sum_{j=1}^{k-1} (f_{j+1}(x) - f_j(x)) \right\} \text{ exists} = f(x).$$

Finally, $f(x) = f_k + \sum_{j=k}^{\infty} (f_{j+1}(x) - f_j(x))$

$$\Rightarrow \|f - f_k\|_p \leq \sum_{j=k}^{\infty} \|f_{j+1} - f_j\|_p \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{1-k} \xrightarrow{k \rightarrow \infty} 0$$

Corollary (Inverse of Dominated Convergence).

If f_n is a sequence of integrable function, and

$$\int_{\mathbb{R}} |f_n - f| d\mu \xrightarrow{n \rightarrow \infty} 0$$

then there exists a subsequence f_{n_k} st

$$f_{n_k} \xrightarrow{k \rightarrow \infty} f(x) \text{ a.e. and } (f_{n_k}(x)) \leq \bar{f} \text{ integrable.}$$

Proof: $f_n \rightarrow f$ in L^1 then f_n is Cauchy sequence in L^1

Then f_{n_k} converges to a function, say g , in two senses:

$$f_{n_k} \rightarrow g \text{ in } L^1 \text{ and } f_{n_k} \rightarrow g \text{ a.e.}$$

But $f_{n_k} \rightarrow f$ in L^1 , we conclude that $f = g$ a.e.

$$\begin{aligned} \text{Furthermore, } |f_{n_k}(x)| &= |f(x) - \sum_{j=k}^{\infty} (f_{j+1}(x) - f_j(x))| \\ &\leq |f(x)| + \sum_{j=k}^{\infty} |f_{j+1}(x) - f_j(x)| \leq 2|f(x)| \end{aligned}$$

and $f \in L^1$.

Ex: Prove that if $f \geq 0$, $\int f < \infty$ then $f < \infty$ a.e.