Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
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Homework Sheet 9 for 17. 12. 2018

9.1. Let $A : D(A) \to H$ be a self-adjoint operator and $A \geq 0$.

(i) Prove that there exist a self-adjoint operator $\sqrt{A} \geq 0$ such that $(\sqrt{A})^2 = A$. Show that $D(\sqrt{A}) = Q(A)$, where $Q(A)$ is the quadratic form domain of A.

(ii) Prove the formula

$$\sqrt{A} = \frac{1}{\pi}\int_0^\infty \frac{A}{(t+A)\sqrt{t}}\mathrm{d}t.$$

9.2. Let $A : D(A) \to H$ be a self-adjoint operator and $A \geq 0$. Assume that the variational problem

$$E = \inf\{\|\sqrt{A}u\|^2 \,:\, u \in D(\sqrt{A}), \|u\| = 1\}$$

has a minimizer $u_0 \in D(\sqrt{A})$. Show that $u_0 \in D(A)$ and $Au_0 = Eu_0$.

9.3. Let $3/2 < s < 2$ and let $A = -\Delta - |x|^{-s}$ on $L^2(\mathbb{R}^3)$ with $D(A) = C_c^\infty(\mathbb{R}^3 \setminus \{0\})$.

(i) Prove that A is bounded from below and the quadratic form domain of A is $Q(A) = H^1(\mathbb{R}^3)$.

(ii) Let A_F be the Friedrichs extension of A. Show that $D(A_F) \neq H^2(\mathbb{R}^3)$.

9.4. Let $A : D(A) \to H$ be a self-adjoint operator. Let $u_0 \in D(A)$ and define

$$u(t) = e^{-itA}u_0, \quad \forall t \in \mathbb{R}.$$

(i) Prove that $u(t) \in D(A)$ for all $t \in \mathbb{R}$.

(ii) Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) := \lim_{s\to t}\frac{u(s)-u(t)}{s-t} = -iAu(t)$$

Here the limit holds strongly in H.

Hint: You can use Spectral theorem and Dominated convergence.