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Mathematical Quantum Mechanics
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Homework Sheet 8 for 10. 12. 2018

8.1. Let $A : D(A) \to H$ be a symmetric operator. Prove that if B is a symmetric closed extension of A, then

$$\overline{A} \subset B \subset A^*.$$

8.2. Let $A : D(A) \to H$ be a symmetric operator. Prove that

$$\mathrm{Rang}(\overline{A} + i) = \overline{\mathrm{Rang}(A + i)}.$$

Here the right side is the closure on H.

8.3. Let $A : D(A) \to H$ be a symmetric operator such that $\sigma(A) \neq \mathbb{C}$. Prove that A is closed, i.e. $A = \overline{A}$.

8.4. Let $1 \le p < q \le \infty$. Prove that for any function $f \in L^r(\mathbb{R}^d)$ with $p \le r \le q$ and for any $\varepsilon > 0$, we can write

$$f = f_1 + f_2$$

with $f_2 \in L^q(\mathbb{R}^d)$, $f_1 \in L^p(\mathbb{R}^d)$ and $\|f_1\|_{L^p} \le \varepsilon$.

Note: This decomposition is useful for applications of Kato–Rellich theorem.

8.5. Let $V : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $A = -\Delta + V$ is a self-adjoint operator on $D(A) = H^2(\mathbb{R}^d)$. Prove that $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$.

8.6. Consider the operator $A = \sqrt{-\Delta} + a|x|^{-1}$ with $D(A) = H^1(\mathbb{R}^3)$ on $L^2(\mathbb{R}^3)$, where $\sqrt{-\Delta}$ is defined by the Fourier transform

$$\widehat{\sqrt{-\Delta} u}(k) = |2\pi k| \widehat{u}(k), \quad \forall u \in H^1(\mathbb{R}^3).$$

Prove that A is a self-adjoint operator if $-1/2 < a < 1/2$.