Homework Sheet 7 for 3. 12. 2018

7.1. Let A be a bounded self-adjoint operator on a Hilbert space H. Let $u \in H$ and let μ_u be the spectral measure associated with A. Prove that

$$\mu_u(\sigma(A)) = \|u\|^2.$$

7.2. Let $A : D(A) \to H$ be a (densely defined) unbounded operator on a Hilbert space H. Prove that the following statements are equivalent:

(i) A is symmetric, i.e. $\langle u, Av\rangle = \langle Au, v\rangle$ for all $u, v \in D(A)$.

(ii) $\langle u, Au\rangle \in \mathbb{R}$ for all $u \in D(A)$.

(iii) A^* is an extension of A.

7.3. Let (Ω, μ) be a measure space. Let $f : \Omega \to \mathbb{C}$ be a measurable function. Prove that the subspace

$$D(M_f) = \{u \in L^2(\Omega, \mu) : fu \in L^2(\Omega, \mu)\}$$

is dense in $L^2(\Omega, \mu)$.

7.4. Let $A : D(A) \to H$ be a self-adjoint operator. Let $1_{(a,b)}$ be the characteristic of the interval $(a, b) \subset \mathbb{R}$, and define the spectral projection $1_{(a,b)}(A)$ by the spectral theorem. Prove that

$$\sigma(A) = \{\lambda \in \mathbb{R} : 1_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) \neq 0 \quad \text{for all } \varepsilon > 0\}.$$

7.5. Let $A : D(A) \to H$ be a self-adjoint operator. Use the Spectral Theorem (multiplication operator version) to prove the Dominated Convergence in functional calculus: if $\{f_n\} \subset B(\sigma(A))$, $f_n(t) \to f(t)$ pointwise and $\sup_n \|f_n\|_{L^\infty} < \infty$, then

$$\|f_n(A)u - f(A)u\| \to 0, \forall u \in H.$$

7.6. For bounded operators on a Hilbert space H, recall the following convergence notions:

- Norm-convergence: $B_n \xrightarrow{n} B$ iff $\|B_n - B\| \to 0$.
- Strong convergence: $B_n \xrightarrow{s} B$ iff $\|B_n u - Bu\| \to 0$ for all $u \in H$.
- Weak convergence: $B_n \xrightarrow{w} B$ iff $\langle u, (B_n - B)v\rangle \to 0$ for all $u, v \in H$.

Find examples for the following cases:

(i) $B_n \xrightarrow{s} B$ strongly, but B_n does not converge to B in operator norm.

(ii) $B_n \xrightarrow{w} B$ weakly, but B_n does not converge to B strongly.