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Mathematical Quantum Mechanics
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Homework Sheet 6 for 26. 11. 2018

6.1. Let $\{x_n\}$ be an orthonormal family in a Hilbert space. Prove that $x_n \rightharpoonup 0$ weakly.

6.2. Let A be a bounded operator on a Hilbert space H. Prove that

$$\mathrm{Ker}(A^*) = \mathrm{Ran}(A)^\perp.$$

Here $\mathrm{Ran}(A) := \{Ax : x \in H\}$ and $\mathrm{Ker}(A^*) := \{x : A^*x = 0\}$.

6.3. Let A be a bounded operator on a Hilbert space H such that there exists $b > 0$:

$$\|Ax\| \geq b\|x\|, \quad \|A^*x\| \geq b\|x\|, \quad \forall x \in H.$$

Show that the inverse A^{-1} exists as a bounded operator and $\|A^{-1}\| \leq b^{-1}$.
Hint: You can prove $\mathrm{Ran}(A) = H$ by showing that $\mathrm{Ran}(A)$ is closed and $\mathrm{Ran}(A)^\perp = \{0\}$.

6.4. Let U be a unitary operator on a Hilbert space H, i.e. $UU^* = U^*U = 1$. Prove that

$$\sigma(U) \subset \{\lambda \in \mathbb{C}, |\lambda| = 1\}.$$

Hint: Problem 6.3 could be helpful.

6.5. Let A be a bounded self-adjoint operator on a Hilbert space H. Prove that

$$\|A\| = \sup_{\|x\|=1} \big|\langle x, Ax\rangle\big|.$$

Hint: You can use $\|A\| = \sup_{\|x\|=\|y\|=1} \mathrm{Re}\langle x, Ay\rangle$ (why?) and polarization identity.

6.6. Consider the operator A on $H = L^2(0,1)$ defined by

$$(Af)(x) = xf(x), \quad \forall f \in H.$$

Prove that A is a bounded self-adjoint operator and $\sigma(A) = [0,1]$, but A has no eigenvalue.