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Mathematical Quantum Mechanics
Winter Semester 2018/19
12. 11. 2018

Homework Sheet 5 for 19. 11. 2018

5.1. Prove that if $f_n \rightharpoonup f$ weakly in $H^1(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3} \frac{|f_n(x)|^2}{|x|}\,\mathrm{d}x \to \int_{\mathbb{R}^3} \frac{|f(x)|^2}{|x|}\,\mathrm{d}x.$$

5.2. Let $S(0,1)$ be the unit sphere in $\mathbb{R}^3$. Prove that for all $x \in \mathbb{R}^3$,

$$\frac{1}{|S(0,1)|}\int_{S(0,1)} \frac{d\omega}{|x-\omega|} = \frac{1}{\max\{|x|,1\}}.$$

Note: This result implies Newton's theorem.

Hint: You can use the mean-value theorem for harmonic functions.

5.3. Let f_n be a sequence in $H^1(\mathbb{R}^d)$. Prove that if $f_n \rightharpoonup f$ and $\nabla f_n \rightharpoonup F$ weakly in $L^2(\mathbb{R}^d)$ then $f \in H^1(\mathbb{R}^d)$ and $\nabla f = F$.

5.4. Let A be a linear operator on a separable Hilbert space H. Prove that the following statements are equivalent:

(i) A is a bounded operator;

(ii) If $x_n \rightharpoonup x$ weakly in H, then $Ax_n \rightharpoonup Ax$ weakly in H.

Hint: You can use Uniform boundedness principle.

5.5. Let A be a bounded operator and let B be a compact operator on a separable Hilbert space H. Prove that AB and BA are compact operators.

5.6. Let A be a bounded self-adjoint operator on a separable Hilbert space H. Assume that the variational problem

$$E = \inf_{\|u\|=1} \langle u, Au\rangle$$

has a minimizer u_0. Prove that $Au_0 = Eu_0$.

Hint: The function $t \mapsto \langle (u_0+tv), A(u_0+tv)\rangle \|u_0+tv\|^{-2}$ is minimized at $t=0$.