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Mathematical Quantum Mechanics
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Homework Sheet 4 for 12. 11. 2018

In this exercise we consider the energy functional of the hydrogen atom

$$\mathcal{E}(u) = \int_{\mathbb{R}^3} |\nabla u(x)|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} \,\mathrm{d}x, \quad u \in H^1(\mathbb{R}^3), \|u\|_{L^2} = 1.$$

4.1. In the lecture we discussed a proof of the stability $\mathcal{E}(u) \geq -C$ using Sobolev's inequality $\|\nabla u\|_{L^2} \geq S\|u\|_{L^6}$ and the decomposition

$$|x|^{-1} \leq R^{-1} + |x|^{-1} 1(|x| \leq R).$$

Work out the details to find an explicit lower bound. You can use the sharp constant

$$S = \frac{\sqrt{3}}{2}(2\pi^2)^{1/3} \approx 2.34.$$

Note: The lower bound obtained in this way is not very good. It will be improved below.

4.2. Show that for all functions $g \geq 0$ satisfying $\int_{\mathbb{R}^3} g^{3/2}(x)\,\mathrm{d}x = S^3$, we have

$$\mathcal{E}(u) \geq \int_{\mathbb{R}^3} |u(x)|^2 \Big(g(x) - |x|^{-1} \Big) \,\mathrm{d}x.$$

Use this with $g(x) = [|x|^{-1} - R^{-1}]_+$ for an appropriate R to deduce that

$$\mathcal{E}(u) \geq -\frac{1}{3}.$$

Note: This bound is remarkably better than $\mathcal{E}(u) \geq -1$ obtained by Hardy's inequality.

4.3. Show that

$$\int_{\mathbb{R}^3} |\nabla u(x)|^2 \,\mathrm{d}x \geq \frac{3}{4} \left(\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} \,\mathrm{d}x \right)^2, \quad \forall u \in H^1(\mathbb{R}^3), \|u\|_{L^2} = 1.$$

Hint: You can use the scaling argument.

4.4. Use the above inequality to prove that for any $s > 0$,

$$\left(\int_{\mathbb{R}^3} |\nabla u(x)|^2 \,\mathrm{d}x \right) \left(\int_{\mathbb{R}^3} |x|^{2s} |u(x)|^2 \,\mathrm{d}x \right)^{1/s} \geq \frac{3}{4}, \quad \forall u \in H^1(\mathbb{R}^3), \|u\|_{L^2} = 1.$$

Note: When $s = 1$ it is Heisenberg's uncertainty principle (with a worse constant).