Homework Sheet 3 for 5. 11. 2018

3.1. Let $2 \le p < \infty$ and $f, g \in L^p(\Omega)$. Prove Clarkson's first inequality

$$\left\| \frac{f+g}{2} \right\|_{L^p(\Omega)}^p + \left\| \frac{f-g}{2} \right\|_{L^p(\Omega)}^p \le \frac{\|f\|_{L^p(\Omega)}^p + \|g\|_{L^p(\Omega)}^p}{2}$$

Note: When $1 < p \le 2$, there is Clarkson's second inequality

$$\left[\left\| \frac{f+g}{2} \right\|_{L^p(\Omega)}^{p/(p-1)} + \left\| \frac{f-g}{2} \right\|_{L^p(\Omega)}^{p/(p-1)} \right]^{p-1} \le \frac{\|f\|_{L^p(\Omega)}^p + \|g\|_{L^p(\Omega)}^p}{2}$$

which is more difficult to prove. (You can try!)

3.2. Let $1 < p < \infty$.

(i) Prove that if $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$, then

$$\liminf_{n\to\infty} \|f_n\|_{L^p} \ge \|f\|_{L^p}.$$

Hint: You can use the dual formula for the L^p norm.

(ii) Prove that if $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ and $\|f_n\|_{L^p} \to \|f\|_{L^p}$, then

$$f_n \to f \quad \text{strongly in } L^p(\Omega).$$

Hint: You can use Clarkson's inequalities.

3.3. Let $f \in L^2(\mathbb{R})$ with compact support.

(i) Prove that the function

$$\widehat{f}(z) = \int_{\mathbb{R}} e^{-2\pi i z x} f(x)\,\mathrm{d}x, \quad z \in \mathbb{C}$$

is an entire function.

(ii) Deduce that if the Fourier transform $\widehat{f}(k)$, $k \in \mathbb{R}$, has compact support, then $f \equiv 0$. (This is a weak form of Hardy's uncertainty principle.)

3.4. Recall the momentum operator $p = -i\nabla_x$ with $x \in \mathbb{R}^d$.

(i) Prove that $[x, p] = x \cdot p - p \cdot x = id$, namely

$$\langle f, [x,p] f \rangle_{L^2(\mathbb{R}^d)} = id \|f\|_{L^2(\mathbb{R}^d)}^2, \quad \forall f \in C_c^1(\mathbb{R}^d).$$

(Here $id = i \times d$ is a complex number, not the "identity".)

(ii) Deduce Heisenberg's uncertainty principle

$$\langle f, p^2 f \rangle_{L^2(\mathbb{R}^d)} \langle f, x^2 f \rangle_{L^2(\mathbb{R}^d)} \ge \frac{d^2}{4} \|f\|_{L^2(\mathbb{R}^d)}^4, \quad \forall f \in C_c^1(\mathbb{R}^d).$$