Homework Sheet 2 for 29. 10. 2018

2.1. Let $1 \le p, q, r \le \infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = \frac{1}{r} + \frac{1}{r'} = 1.$$

Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ and $h \in L^r(\mathbb{R}^d)$.

(i) Prove Young 's inequality

$$\Big| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(y) h(x-y) \mathrm{d}x \mathrm{d}y \Big| \le \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$

Hint: Use Hölder's inequality for $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \alpha^{1/r'} \beta^{1/p'} \gamma^{1/q'}$ where

$$\alpha = |f(x)|^p |g(y)|^q, \beta = |g(y)|^q |h(x-y)|^r, \gamma = |f(x)|^p |h(x-y)|^r.$$

(ii) Deduce that

$$\|f * g\|_{L^{r'}} \le \|f\|_{L^p} \|g\|_{L^q}.$$

2.2. Prove that

$$\widehat{e^{-\pi|x|^2}}(k) = e^{-\pi|k|^2} \quad \text{on } \mathbb{R}^d.$$

Hint: You may reduce to $d = 1$ and show that $\frac{\mathrm{d}}{\mathrm{d}k} \int_{\mathbb{R}} e^{-\pi(x+ik)^2} \mathrm{d}x = 0$.

2.3. Let $1 \le p < q < r \le \infty$. Let $\{f_n\}$ be a sequence in $L^p(\Omega) \cap L^r(\Omega)$ such that

$$\exists \varepsilon > 0 : \|f_n\|_{L^p} \le 1, \ \|f_n\|_{L^r} \le 1 \text{ and } \|f_n\|_{L^q} \ge \varepsilon \text{ for all } n \ge 1.$$

Prove that there exists a constant $\delta > 0$ such that

$$\liminf_{n \to \infty} \mu(\{x : |f_n(x)| > \delta\}) \ge \delta.$$

Hint: You can use the Layer–cake representation for the norms.

2.4. In this problem we consider the relation of strong and weak convergences.

(i) Let $1 \le p < q < \infty$ and let $\{f_n\}$ be a bounded sequence in $L^p(\Omega) \cap L^q(\Omega)$. Prove that if $f_n \to f$ strongly in $L^p(\Omega)$, then $f_n \to f$ strongly in $L^s(\Omega)$ for all $p \le s < q$ and $f_n \rightharpoonup f$ weakly in $L^q(\Omega)$.

(ii) Find a sequence $\{f_n\}$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $f_n \to 0$ strongly in $L^1(\mathbb{R})$ and $f_n \rightharpoonup 0$ weakly in $L^2(\mathbb{R})$ but f_n does not converge strongly in $L^2(\mathbb{R})$.