Homework Sheet 14 for 4. 2. 2019

14.1. Let $\{u_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $L^2(\Omega_1, \mathrm{d}\mu_1)$ and let $\{v_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for $L^2(\Omega_2, \mathrm{d}\mu_2)$. Prove that $\{u_i \otimes v_j\}_{i,j\in\mathbb{N}}$ is an orthonormal basis for $L^2(\Omega_1 \times \Omega_2)$.

14.2. Prove that for any $N \geq 1$ and $R > 0$, there exists a partition of unity

$$\sum_{j=0}^{N} \varphi_j^2 = 1$$

where $\varphi_j : \mathbb{R}^{3N} \to [0, 1]$ satisfy all of the following properties

i) $\operatorname{supp}\varphi_0 \subset \{x = (x_1, ..., x_N) \in (\mathbb{R}^3)^N : \max_{1\leq i\leq N} |x_i| \leq 2R\}$;

ii) $\operatorname{supp}\varphi_j \subset \{x = (x_1, ..., x_N) \in (\mathbb{R}^3)^N : |x_j| \geq R\}$, for all $j = 1, 2, ..., N$;

iii) $\displaystyle\sum_{j=0}^{N} |\nabla\varphi_j| \leq CR^{-1}$, where $C > 0$ is independent of R.

14.3. Consider the operator

$$H_N = \sum_{i=1}^{N} (-\Delta_{x_i} - |x_i|^{-1}), \quad x = (x_1, x_2, ..., x_N) \in (\mathbb{R}^3)^N.$$

We know that H_N is a self-adjoint on the anti-symmetric space $L^2_a((\mathbb{R}^3)^N)$ with domain $H^2_a((\mathbb{R}^3)^N)$. Prove that the ground state energy E_N of H_N satisfies

$$E_N = -\frac{(3N)^{1/3}}{4} + o(N^{1/3})_{N\to\infty}.$$

Hint: You can use the fact (without proof) that all eigenvalues of the hydrogen operator $-\Delta - |x|^{-1}$ on $L^2(\mathbb{R}^3)$ are $-1/(4n^2)$ with multiplicity n^2.

14.4. Let $a > 0$. Consider the "scattering problem of hard sphere"

$$E = \inf\left\{ \int_{\mathbb{R}^3} |\nabla f(x)|^2 \,\mathrm{d}x : 0 \leq f \leq 1, f(x) = 0 \text{ if } |x| < a \text{ and } f(x) \to 1 \text{ as } |x| \to \infty \right\}.$$

Show that $E = 4\pi a$ and $f(x) = 1 - a/|x|$ is the unique minimizer.

Hint: You may first consider radial functions.

14.5. Let $a > 0$. Consider the Gross-Pitaevskii functional

$$\mathcal{E}_{\mathrm{GP}}(u) = \int_{\mathbb{R}^3} (|\nabla u(x)|^2 + |x|^2|u(x)|^2 + 4\pi a|u(x)|^4) \,\mathrm{d}x.$$

Prove that the variational problem

$$E_{\mathrm{GP}} = \inf\{\mathcal{E}_{\mathrm{GP}}(u) : u \in H^1(\mathbb{R}^3), \|u\|_{L^2(\mathbb{R}^3)} = 1\}$$

has a minimizer u_0 and it satisfies the equation

$$-\Delta u_0(x) + |x|^2 u_0(x) + 8\pi a|u_0(x)|^2 u_0(x) = \mu u_0(x)$$

for a constant $\mu \in \mathbb{R}$. Deduce that $u_0 \in C^\infty(\mathbb{R}^3)$.