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Mathematical Quantum Mechanics
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Homework Sheet 13 for 28. 1. 2019

13.1. Let $\{u_n\}$ be an orthonormal basis of a Hilbert space H. Prove that

i) If K is a trace class operator on H, then $\sum_{n=1}^{\infty} |\langle u_n, K u_n\rangle|$ is finite and $\sum_{n=1}^{\infty}\langle u_n, K u_n\rangle$ is independent of the choice of $\{u_n\}$.

ii) If K is a Hilbert–Schmidt operator on H, then $\sum_{n=1}^{\infty} \|K u_n\|^2$ is finite and independent of the choice of $\{u_n\}$.

13.2. Let $0 \le V \in L^{3/2}(\mathbb{R}^3)$. Consider the operator $K = \sqrt{V}(-\Delta)^{-1}\sqrt{V}$ (with $\sqrt{V}$ being the multiplication operator). Show that K is a Hilbert-Schmidt operator and

$$\|K\|_{\mathrm{HS}} \le C\|V\|_{L^{3/2}}$$

for a universal constant $C > 0$ independent of V.
Hint: You can compute the integral kernel of K (the kernel of $(-\Delta)^{-1}$ is $(4\pi|x-y|)^{-1}$).

13.3. We prove the Lieb–Thirring inequality for the sum of negative eigenvalues of $-\Delta+V$ by adapting the proof of the CLR inequality.

i) Let $\{u_n\}_{n=1}^{N}$ be an orthonormal family in $L^2(\mathbb{R}^3)$ with $u_n \in H^2(\mathbb{R}^3)$ and denote $\rho(x) = \sum_{n=1}^{N} |u_n(x)|^2$. Prove that there exists a universal constant $K > 0$ such that

$$\sum_{n=1}^{N} \int_{\mathbb{R}^3} |\nabla u_n(x)|^2 \,\mathrm{d}x \ge K \int_{\mathbb{R}^3} \rho(x)^{5/3} \,\mathrm{d}x.$$

ii) Deduce that if $V \in L^{5/2}(\mathbb{R}^3, \mathbb{R})$, then the sum of negative eigenvalues of $-\Delta + V$ is bounded from below by $-C \int_{\mathbb{R}^3} |V|^{5/2}$ with a universal constant $C > 0$ (independent of V). Here recall that $-\Delta + V$ is a self adjoint operator on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$.

13.4. Let $V \in L^p(\mathbb{R}^3)$ with $2 \le p < 3$. Show that for all $u \in C_c^\infty(\mathbb{R}^3)$,

$$\int_1^\infty \|V e^{it\Delta} u\|_{L^2(\mathbb{R}^3)} \mathrm{d}t < \infty.$$

Note: This bound implies the existence of wave operator for $-\Delta + V$.

13.5. Let A and B be self-adjoint operators on a Hilbert space H such that the wave operator

$$\Omega := \lim_{t\to\infty} e^{itA} e^{-itB}$$

exists, i.e. the strong limit $e^{itA}e^{-itB}u \to \Omega u$ holds for all $u \in H$. Prove that

i) $A\Omega = \Omega B$ (i.e. $A\Omega u = \Omega B u$ for all $u \in D(B)$).

ii) If A has no eigenvalue, then B also has no eigenvalue.