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Mathematical Quantum Mechanics
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Homework Sheet 12 for 21. 1. 2019

12.1. Let $A : D(A) \to H$ be a self-adjoint operator which is bounded from below. Let $\{B_m\}$ be a sequence of bounded self-adjoint operator such that $\|B_m\| \to 0$ as $m \to \infty$. Prove the convergence of the min-max values

$$\lim_{m\to\infty} \mu_n(A + B_m) = \mu_n(A), \quad \forall n \geq 1.$$

12.2. Let $A : D(A) \to H$ be a self-adjoint operator which is bounded from below. Assume that its min-max values satisfy $\mu_n(A) \to \infty$ as $n \to \infty$. By the min-max principle, we know that $\{\mu_n\}_{n=1}^\infty$ are eigenvalues and we can choose an orthonormal *family* of eigenfunctions $\{u_n\}_{n=1}^\infty$. Prove that $\{u_n\}_{n=1}^\infty$ is an orthonormal *basis* for H.

12.3. Let $A : D(A) \to H$ be a self-adjoint operator and let $u \in H$ be a normalized vector. Prove that if $A \geq 0$, then the operator

$$A - |u\rangle\langle u|$$

has at most one negative eigenvalue.

12.4. Let $V \in L^{3/2}(\mathbb{R}^3)$. We know that the operator $A = -\Delta + V(x)$ is bounded from below (e.g. on the core domain $C_c^\infty(\mathbb{R}^3)$) and hence it can be extended to be a self-adjoint operator on $L^2(\mathbb{R}^3)$ by Friedrichs' method. Show that its min-max values satisfy $\mu_n \leq 0$ and $\mu_n \to 0$ as $n \to \infty$.

Hint: Note that here we are not assuming $V \in L^2 + L^p$ with $2 \leq p < \infty$, so we cannot deduce that $\sigma_{\text{ess}}(A) = [0, \infty)$ from Weyl's theory in the lecture.

12.5. Let $V \in C_c^\infty(\mathbb{R}^3)$ such that $V \leq 0$ and $V \not\equiv 0$. For any $\lambda > 0$, denote by N_λ the number of negative eigenvalues of $-\Delta + \lambda V$ (which is a self-adjoint operator on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$). Prove that $N_\lambda \to 0$ as $\lambda \to 0$ and $N_\lambda \to \infty$ as $\lambda \to \infty$.

12.6. Let $a > 0$ and $R > 0$. Let $u \in L^2(\mathbb{R}^d)$ satisfy that

$$a^2 \int_{\mathbb{R}^d} \varphi(x)^2 |u(x)|^2 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla \varphi(x)|^2 |u(x)|^2 \, \mathrm{d}x$$

for all functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi, \nabla\varphi \in L^\infty$ and $\text{supp}\varphi \subset \{|x| \geq R\}$. Prove that

$$\int_{\mathbb{R}^d} e^{2b|x|} |u(x)|^2 \, \mathrm{d}x < \infty, \quad \forall b < a.$$

Hint: Think of the choice $\varphi = e^f - 1$. This result was used in the lecture to prove the decay of bound states of Schrödinger operator.