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Mathematical Quantum Mechanics
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# **Homework Sheet 11** for 14. 1. 2019

**11.1.** Let $A$ be a self-adjoint compact operator on an infinitely dimensional Hilbert space. Prove that $\sigma_{\text{ess}}(A) = \{0\}$.

**11.2.** Let $V : \mathbb{R}^3 \to \mathbb{R}$ satisfy that $V \in L^2(\mathbb{R}^3) + L^p(\mathbb{R}^3)$ with $2 \leq p < \infty$ and

$$V(x) \leq -\frac{1}{|x|^s}$$

for a constant $0 < s < 2$. Recall by Kato-Rellich theorem we know that $A = -\Delta + V$ is a self-adjoint operator on $L^2(\mathbb{R}^3)$ with domain $D(A) = H^2(\mathbb{R}^3)$. Prove that $A$ has infinitely many *negative eigenvalues*.

**11.3.** Let $A : D(A) \to H$ be a self-adjoint operator and let $u_0$ be a normalized eigenfunction of $A$, i.e. $\|u_0\| = 1$ and $Au_0 = \lambda u_0$ for some $\lambda \in \mathbb{R}$. Prove that the Schrödinger equation

$$iu'(t) = Au(t) \text{ for all } t \in \mathbb{R}, \quad u(0) = u_0$$

has a unique solution

$$u(t) = e^{-it\lambda} u_0, \quad \forall t \in \mathbb{R}.$$

**11.4.** Let $A$ be a self-adjoint operator on $L^2(\mathbb{R}^d)$ and consider the Schrödinger equation

$$i\partial_t u(t,x) = Au(t,x) \text{ for all } t \in \mathbb{R}, \quad u(0,x) = u_0(x)$$

where $u_0$ belongs to the span of eigenfunctions of $A$, i.e. $u_0 = \sum_{n=1}^N v_n$ with $\{v_n\}$ an orthogonal family of eigenfunction of $A$. Prove that for any $\varepsilon > 0$, there exists $R = R_\varepsilon$ such that

$$\int_{|x| \geq R} |u(t,x)|^2 \, \mathrm{d}x \leq \varepsilon, \quad \forall t \in \mathbb{R}.$$

This means that $u(t, \cdot)$ is localized in space for all time.