Mathematisches Institut der LMU
Prof. P. T. Nam, Prof. A. Scrinzi
D.T. Nguyen

Mathematical Quantum Mechanics
Winter Semester 2018/19
16. 10. 2018

Homework Sheet 1 for 22. 10. 2018

1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function (i.e. $f(x) \le f(y)$ if $x \le y$).

(i) Prove that for every $x \in \mathbb{R}$ the following limits exist

$$f_+(x) = \lim_{y \downarrow x} f(y), \quad f_-(x) = \lim_{y \uparrow x} f(y)$$

(ii) Show that f is *not* continuous at x if and only if $f_+(x) > f_-(x)$.

(iii) Deduce that f is continuous for all $x \in \mathbb{R}$ except a countable set.

Remark: The property (iii) has been used to define the Lebesgue integral via the Riemann integral (with the measures of level sets).

1.2. In this exercise we prove the Brezis-Lieb refinement for Fatou's lemma in $L^p(\Omega, d\mu)$. Let $1 < p < \infty$. Let $\{f_n\}_{n=1}^\infty \subset L^p(\Omega)$ satisfy $f_n(x) \to f(x)$ for a.e. $x \in \Omega$ as $n \to \infty$ and $\|f_n\|_{L^p} \le C$ (uniformly in n).

(i) Prove that for all $a, b \in \mathbb{C}$ and for all $0 < \lambda < 1$,

$$|a+b|^p \le \lambda^{1-p}|a|^p + (1-\lambda)^{1-p}|b|^p.$$

Hint: Use the convexity of $a \mapsto |a|^p$.

(ii) Deduce from (i) that for all $\varepsilon > 0$, there exists $C_{\varepsilon,p} > 0$ (depending only on ε and p) such that we have the pointwise inequality

$$||f_n(x)|^p - |f_n(x) - f(x)|^p| \le \varepsilon |f_n(x) - f(x)|^p + C_{\varepsilon,p}|f(x)|^p.$$

(iii) Prove that

$$\int_\Omega \Big| |f_n(x)|^p - |f(x)|^p - |f_n(x) - f(x)|^p \Big| d\mu(x) \to 0, \quad \text{as } n \to \infty.$$

Hint: Use Dominated Convergence for $G_{n,\epsilon} = \Big(\big| |f_n|^p - |f|^p - |f_n - f|^p \big| - \varepsilon |f_n - f|^p \Big)_+$.

1.3. Let $f \in C_c^\infty(\mathbb{R}^d)$ (infinitely smooth with compact support) and $g \in L_c^1(\mathbb{R}^d)$ (integrable with compact support). Prove that the convolution $f * g$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y) \, \mathrm{d}y, \quad \forall x \in \mathbb{R}^d$$

belongs to $C_c^\infty(\mathbb{R}^d)$.