Mathematisches Institut der LMU
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# Mathematical Quantum Mechanics

Final Exam

Nachname: ____________ Vorname: ____________

Geburtstag: ____________ Matrikelnr.: ____________

Studiengang: ____________ Fachsemester: ____________

Please place your identity and student ID cards on the table. Switch off all electronic devices. You can use your notes.

Please write your name and your solutions on the provided sheets. Justify all your statements by either proving them directly or referring to the material discussed in class.

You have 180 minutes. Good luck!

| Problems | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Maximum points | 15 | 15 | 20 | 20 | 20 | 10 |
| Scored points | | | | | | |

| Final exam | Homework bonus | Midterm bonus | Total points | GRADE |
|---|---|---|---|---|
| | | | | |

**Problem 1.** Let $A$ be a self-adjoint operator on a separable Hilbert space $\mathscr{H}$. Prove that the following statements are equivalent:

$$(A+i)^{-1} \text{ is compact} \iff (A^2+1)^{-1/2} \text{ is compact} \iff (A^2+1)^{-s} \text{ is compact } \forall s>0.$$

**Problem 2.** Consider the free Schrödinger dynamics $u(t) = e^{it\Delta}u_0$ on $L^2(\mathbb{R}^3)$, $t \in \mathbb{R}$. Prove that if *either* $u_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ *or* $u_0 \in H^2(\mathbb{R}^3)$, then

$$\lim_{|t|\to\infty} \int_{\mathbb{R}^3} \frac{|u(t,x)|^2}{|x|^s} dx = 0, \quad \forall 0 < s < 3.$$

**Problem 3.** Consider the operator $A$ on $L^2(0,1)$ defined by

$$(Af)(x) = -(x^{2020} f'(x))' - x\int_0^1 y f(y)dy, \quad D(A) = C_c^\infty(0,1).$$

(a) Prove that $A$ is symmetric and bounded from below.

(b) Prove that $A_F$, the Friedrichs' extension of $A$, has *at most* one negative eigenvalue.

**Problem 4.** Consider the operator $A$ on $L^2(\mathbb{R}^3)$ defined by

$$(Au)(x) = -\Delta u(x) + \int_{\mathbb{R}^3} e^{-\pi|x-y|^2} u(y)dy, \quad D(A) = H^2(\mathbb{R}^3).$$

(a) Compute $\mathscr{F}A\mathscr{F}^*$, with $\mathscr{F}$ the Fourier transform. Deduce that $A$ is self-adjoint and has spectrum $\sigma(A) = [1,\infty)$.

(b) Consider $B = A + |x|^2$ with $D(B) = C_c^\infty(\mathbb{R}^3)$. Prove that $B_F$, the Friedrichs' extension of $B$, has compact resolvent.

**Problem 5.** Let $\lambda > 0$ and consider the operator $A$ on $L^2(\mathbb{R}^3)$ defined by

$$A = -\Delta - \frac{e^{-\lambda|x|}}{|x|+\lambda}, \quad D(A) = H^2(\mathbb{R}^3).$$

(a) Prove that $A$ is self-adjoint and its number of negative eigenvalues is bounded by $C\min\{\lambda^{-3/2}, \lambda^{-9/2}\}$ for a universal constant $C > 0$.

(b) Consider the asymptotic completeness:

"The wave operators $\lim_{t\to\pm\infty} e^{itA}e^{it\Delta}$, $\lim_{t\to\pm\infty} e^{it\Delta}e^{itA}$ exist on $L^2(\mathbb{R}^3)$".

Prove that it holds for $\lambda$ sufficiently large, but fails for $\lambda$ sufficiently small.

**Problem 6.** Consider the operator $A$ on $L^2(\mathbb{R})$ (c.f. Homework 7.2 & Midterm exam)

$$(Af)(x) = (1+x^2)f(x), \quad D(A) = \left\{ f \in L^2(\mathbb{R}) : (1+x^2)f \in L^2(\mathbb{R}), \int_{\mathbb{R}} f = 0 \right\}.$$

Prove that $A$ has *at least* two different self-adjoint extensions.