Mathematisches Institut der LMU
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Mathematical Quantum Mechanics
Winter Semester 2018-2019
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Mathematical Quantum Mechanics

Final Exam

Nachname: ____________________ Vorname: ____________________

Geburtstag: ____________________ Matrikelnr.: ____________________

Studiengang: ____________________ Fachsemester: ____________________

INSTRUCTIONS:

Please place your identity and student ID cards on the table so that they are clearly visible. Switch off your mobile phone and all other electronic devices.

Please write your name on every sheet. Prove all your statements or refer to the results discussed in class. You can use your notes. You can try any problem and collect partial credits.

You have 180 minutes. Good luck!

Problems	1	2	3	4	5	$\sum$
Maximum points	15	20	20	20	25	100
Scored points						

Homework bonus		Midterm bonus		Total points		FINAL GRADE	

Name:

Problem 1. Let $Z > 1$ and consider the Hartree functional

$$\mathscr{E}(u) = \int_{\mathbb{R}^3} |\nabla u(x)|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} \frac{Z|u(x)|^2}{|x|} \,\mathrm{d}x + \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} \,\mathrm{d}x \,\mathrm{d}y.$$

(i) (5 points) Prove that the ground state energy

$$E = \inf \left\{ \mathscr{E}(u) \,:\, u \in H^1(\mathbb{R}^3), \|u\|_{L^2(\mathbb{R}^3)} = 1 \right\}$$

satisfies

$$-\frac{Z^2}{4} \leq E \leq -\frac{(Z-1)^2}{4}.$$

(ii) (10 points) Given that E has a minimizer $u_0 \in H^1(\mathbb{R}^3)$, which solves the equation

$$-\Delta u_0(x) - \frac{Z}{|x|} u_0(x) + 2(|u_0|^2 * |\cdot|^{-1})(x) u_0(x) = \mu u_0(x)$$

in the distributional sense (with a constant $\mu \in \mathbb{R}$). Prove that $u_0 \in H^2(\mathbb{R}^3)$.

Solution:

Name:

Problem 2. We know that the operator $A = -\Delta + |x|^2$ can be defined as a self-adjoint operator on $L^2(\mathbb{R}^3)$ with domain $D(A) \supset C_c^\infty(\mathbb{R}^3)$ by Friedrichs' method.

(i) (5 points) Prove that $A \geq 3$.

(ii) (5 points) Prove that $A^2 - |\Delta|^2 - |x|^4$ is bounded from below.

(iii) (10 points) Prove that the multiplication operator $V(x) = |x|$ is A-relatively compact.

Solution:

Name:

Problem 3. Let A be a positive trace class operator on a separable Hilbert space H with $\mathrm{Tr}[A] = 1$.

(i) (5 points) Let $\mu_k(A)$ be the k-th largest eigenvalue of A (i.e. $-\mu_k(A)$ is the k-th min-max value of $-A$). Prove that for any projection P, we have

$$0 \leq \mu_k(PAP) \leq \mu_k(A), \quad \forall k \in \mathbb{N}.$$

(ii) (5 points) Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of projections such that $\|P_n u\| \to 0$ for all $u \in H$. Prove that

$$\lim_{n\to\infty} \mathrm{Tr}[P_n A P_n] = 0.$$

(iii) (10 points) Prove that if the entropy $S(A) = -\mathrm{Tr}[A \log A]$ is finite, then

$$\lim_{n\to\infty} S(P_n A P_n) = 0$$

with the projections $\{P_n\}$ as in (ii). Could the condition $S(A) < \infty$ be relaxed?

Solution:

Name:

Problem 4. We know that for any $\lambda > 0$, $A = -\Delta - e^{-\lambda|x|}$ is a self-adjoint operator on $L^2(\mathbb{R}^3)$ with domain $D(A) = H^2(\mathbb{R}^3)$ and $\sigma_{\text{ess}}(A) = [0, \infty)$.

(i) (10 points) Let N_λ be the number of negative eigenvalues of A. Prove that

$$N_\lambda \le C\lambda^{-3} \text{ for all } \lambda > 0 \text{ (with } C \text{ independent of } \lambda) \quad \text{and} \quad N_\lambda \to \infty \text{ as } \lambda \to 0.$$

(ii) (10 points) Prove that if λ is large enough, then A has no eigenvalue.

Solution:

Name:

Problem 5. Let $g \in C_c^\infty(\mathbb{R}^3)$ and consider the operator

$$A = -\Delta - \frac{1}{1+4|x|^2} - |g\rangle\langle g|.$$

(i) (5 points) Prove that A is a self-adjoint operator on $L^2(\mathbb{R}^3)$ with domain $D(A) = H^2(\mathbb{R}^3)$ and $\sigma_{\text{ess}}(A) = [0, \infty)$.

(ii) (10 points) Prove that the following strong limit exists for all $u \in L^2(\mathbb{R}^3)$

$$\lim_{t\to\infty} e^{-itA} e^{it(-\Delta)} u.$$

(iii) (10 points) Let E_N^b (or E_N^f) be the ground state energy of N bosons (or fermions) with Hamiltonian $\sum_{k=1}^N A_{x_k}$. Prove that for all $N \in \mathbb{N}$,

$$E_N^b \geq -N\|g\|^2_{L^2(\mathbb{R}^3)} \quad \text{and} \quad E_N^f \geq -\|g\|^2_{L^2(\mathbb{R}^3)}.$$

Solution: