The truncation method for a two-dimensional nonhomogeneous backward heat problem

Phan Thanh Nam[1], Dang Duc Trong[2] and Nguyen Huy Tuan[3*]

[1]Department of Mathematical Sciences, University of Copenhagen, Denmark.

[2]Faculty of Mathematics and Computer Sciences, University of Science, Vietnam National University, HoChiMinh city, Vietnam.

[3]Faculty of Mathematics , SaiGon University, HoChiMinh city, Vietnam.

Abstract

We consider the backward heat problem

$$
\begin{aligned}
&u_t - u_{xx} - u_{yy} = f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T),\\
&u(x,y,T) = g(x,y), \quad (x,y) \in \Omega,
\end{aligned}
$$

with the homogeneous Dirichlet condition on the rectangle $\Omega = (0,\pi) \times (0,\pi)$, where the data f and g are given approximately. The problem is severely ill-posed. Using the truncation method for Fourier series we propose a simple regularized solution which not only works on a very weak condition on the exact data but also attains, due to the smoothness of the exact solution, explicit error estimates which include the approximation $(\ln(\epsilon^{-1}))^{3/2}\sqrt{\epsilon}$ in $H^2(\Omega)$. Some numerical examples are given to illuminate the effect of our method.

Keywords and phrases: Backward heat problem, Ill-posed problem, Nonhomogeneous heat, truncation method, error estimate.

Mathematics subject Classification 2000: 35K05, 35K99, 47J06, 47H10.

*Corresponding author. E-mail: tuanhuy_bs@yahoo.com

1 Introduction

Let $\Omega = (0,1)\times(0,1)$ be a heat conduction rectangle. Given the heat source $f(x,y,t)$ on $(x,y,t) \in \Omega \times [0,T]$ and the final temperature $u(x,y,T)$ at some time $T > 0$, we consider the problem of recovering the temperature distribution $u(x,y,t)$ from the backward heat problem

$$u_t - u_{xx} - u_{yy} = f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T), \tag{1}$$

$$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0, \quad t \in (0,T), \tag{2}$$

$$u(x,y,T) = g(x,y), \quad (x,y) \in \Omega. \tag{3}$$

Since f and g come from measurement, they are in general non-smooth and only approximate values. This is a typical example of the inverse and ill-posed problem since although this problem has at most one solution (see Theorem 1 in Section 2), the solution does not always exist, and in the case of existence, it does not depend continuously on the given data. The instability makes the numerically calculus difficult and hence a regularization is in order.

The homogeneous backward heat problems, i.e. the case $f = 0$, was extensively considered by many authors using many approach, e.g. the original quasi-reversibility method of Lattes and Lions [10], the quasi-boundary value problem method [15], the quasi-solution method of Tikhonov [16], the logarithmic convexity method [1] and the C-regularized semi-groups technique [7]. Physically, this problem arises from the requirement of recovering the heat temperature at some earlier time using the knowledge about the final temperature. The problem is also involved to the situation of a particle moving in a environment with constant diffusion coefficient (see [6]) when one asks to determine the particle position history from its current place. The interest of backward heat equations also comes from financial mathematics, where the celebrated Black-Scholes model [2] for call option can be transformed into a backward parabolic equation whose form is related closely to backward heat equations. Although there are many papers on the homogeneous backward heat equation, the result on the inhomogeneous case is very scarce while the inhomogeneous case is, of course, more general and nearer to practical application than the homogeneous one. Shortly, it allows the appearance of some heat source which is inevitable in nature.

Let us mention here some approachs and their technical difficulties of many earlier works. In the method of quasi-reversibility, the main ideas is of replacing the unbounded operator A (in our case is $-\Delta$) by a perturbed one A_ϵ. In the original method in 1967, Lattes and Lions [25] proposed $A_\epsilon(A) = A - \epsilon A^*A$, i.e. adding a "corrector" into the original operator, to obtain a well-posed problem. The essential difficulty of the quasi-reversibility method is due to the appearance of the second-order operator A^*A which produces serious difficulties on the numerical implementation. In addition, the stability magnitude of the approximating problem, i.e. the error introduced by a small change in the final value, is of order $e^{\frac{T}{\epsilon}}$ which is very large when ϵ becomes small.

In 1983, Showalter [15] presented the quasi-boundary value method for the homogeneous problem which gave a stability estimate better than the one of the quasi-reversibility method discussed above. The main ideas of this method is of adding an appropriate "corrector" into the final data (instead of the main equation). Using this method, Clark and Oppenheimer [3], and very recently Denche and Bessila [4], regularized the backward heat problem by replacing the final condition by

$$u(T) + \epsilon u(0) = g$$

and

$$u(T) - \epsilon u'(0) = g,$$

respectively. This method, in general, gives the stability estimate of order ϵ^{-1}.

Although there are many papers on the homogeneous case of the backward problem, we only find a few result on the inhomogeneous case, and especially the two dimensional case is very scarce. In 2006, Trong and Tuan [17] approximated a one dimensional inhomogeneous linear problem by the quasi-reversibility method. As we mention before, the stability magnitude of the method is of order $e^{\frac{T}{\epsilon}}$. In their work the error between the approximate problem and the exact solution is

$$\epsilon(T-t)\sqrt{\frac{8}{t^4}\|u(.,0)\|^2 + t^2\|\frac{\partial^4 f(x,t)}{\partial x^4}\|^2_{L^2(0,T;L^2(0,\pi))}},$$

which is very large when t becomes small. In 2007, Trong et al. [19] used an improved version of quasi boundary value method to regularize the one-dimensional version of (1-3) for a nonlinear heat source $f = f(x,t,u)$. Their error estimate is $\epsilon^{t/T}$ for $t > 0$ and $(\ln(1/\epsilon))^{1/4}$ for $t = 0$.

One of the essential requirements of the previous works on inhomogeneous problem, e.g. [17, 19], is

$$\sum_{k=1}^{\infty} e^{2Tk^2} g_k^2 < \infty \tag{4}$$

where g_k is the coefficient of the Fourier series of the final datum $u(.,T) = g$, i.e.

$$g_k = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(kx) dx.$$

While such a condition is reasonable in homogeneous problems, it is not necessarily true in the inhomogeneous case. For example, consider the problem

$$\begin{aligned} &u_t - u_{xx} = f(x,t) \equiv e^t x, \quad (x,t) \in (0,\pi) \times (0,T), \\ &u(0,t) = u(\pi,t) = 0, \quad t \in (0,T). \end{aligned}$$

Corresponding to the final value $u(x,1) = g(x) \equiv ex$, the equation has a (unique) solution $u(x,t) = e^t x$. However, by direct computation we find that $g_k = 2e\frac{(-1)^{k+1}}{k}$ and hence

$$\sum_{k=1}^{\infty} e^{2Tk^2} g_k^2 = 4e^2 \sum_{k=1}^{\infty} \frac{e^{2Tk^2}}{k^2} > \infty.$$

In the present paper, we do not need condition (4). In fact, we shall give a simple and convenient way to construct the regularized method which works with very weak assumption on the exact solution.

Let us give a simple analysis for the ill-posedness of the problem(1-3). This problem may be rewritten formally as

$$u(x,y,t) = \sum_{m,n=1}^{\infty} e^{(T-t)(m^2+n^2)} \left(g_{mn} - \int\limits_t^T e^{(s-T)(m^2+n^2)} f_{mn}(s) ds \right) \sin(mx) \sin(ny) \tag{5}$$

where g_{mn} and $f_{nm}(t)$ are the coefficient of the Fourier-sin expansion of g and $f(.,.,t)$, i.e.

$$\begin{aligned} g_{mn} &:= \frac{4}{\pi^2} \int_{\Omega} g(x,y) \sin(mx) \sin(ny) dx dy, \\ f_{mn}(t) &:= \frac{4}{\pi^2} \int_{\Omega} f(x,y,t) \sin(mx) \sin(ny) dx dy. \end{aligned}$$

If $t < T$ then $e^{(T-t)(m^2+n^2)}$ increases very fast when $m^2 + n^2$ becomes large. Thus the term $e^{-(t-T)(m^2+n^2)}$ is the source of instability.

It is a natural think to recover the stability of problem (5) is to filter all high frequencies. In the present paper, we simply do that by using the truncated regularization method, namely taking the sum (5) only for $m^2+n^2 \le M_\epsilon$ with an appropriate regularization parameter M_ϵ. The truncated regularization method is a very simple and effective method for solving some ill-posed problems and it has been successfully applied to some inverse heat conduction problems [5, 8, 13]. However, in many earlier works, we find that only logarithmic type estimates in L^2-norm are available; and estimates of Hölder type are very rate (see Remark 5 and Remark 6 for more detail comparisons). In our method, corresponding to different levels of the smoothness of the exact solution, the convergence rates will be improved gradually. In particular, if we impose a condition similar to (4) then the error estimate in $H^2(\Omega)$ is $(\ln(\epsilon))^{3/2}\sqrt{\epsilon}$, which is better than any Hölder estimate of order ϵ^q with $q \in (0, 1/2)$. We mention that our regularized solution in all case is unique, and all error estimates are valid for all $t \in [0, T]$.

The remainder of the paper is organized as follows. In Section 2 we shall construct the regularized and show that it works even with very weak condition on the exact solution. In Section 3, many error estimates are derived, in both of the usual cases such as the exact solution u in $H_0^1(\Omega)$ or $H^2(\Omega)$, and the special cases when the exact solution is very smooth. Some numerical experiments are given in Section 4 to illuminate the effect of our method.

2 Regularized solution

Let us first make clear what a weak solution of the problem (1-3) is. As follows we shall write $u(t) = u(.,.,t)$ for short. We call a function $u \in C\left([0,T]; L^2(\Omega)\right) \cap C^1((0,T); L^2(\Omega))$ to be a weak solution for the problem (1-3) if

$$\frac{d}{dt}\left\langle u(t), W\right\rangle_{L^2(\Omega)} - \left\langle u(t), \Delta W\right\rangle_{L^2(\Omega)} = \left\langle u(t), \Delta W\right\rangle_{L^2(\Omega)}, \tag{6}$$

for all function $W(x, y) \in H^2(\Omega) \cap H_0^1(\Omega)$. In fact, it is enough to choose W in the orthogonal basis $\{\sin(mx)\sin(ny)\}_{m,n\ge 1}$ and the formula (6) reduces to

$$u_{mn}(t) = e^{(T-t)(m^2+n^2)} g_{mn} - \int_t^T e^{(s-t)(m^2+n^2)} f_{mn}(s) ds, \quad \forall m, n \ge 1 \tag{7}$$

which may also be written formally as (5).

Note that if the exact solution u is smooth then the exact data (f, g) is smooth also. However, the real data, which come from practical measure, is often discrete and non-smooth. We shall therefore always assume that $f \in L^1(0, T; L^1(\Omega))$ and $g \in L^1(\Omega)$, and the error of the data is given on L^1 only. Note that (7) still makes sense with such data, and this formula gives immediately the uniqueness.

Theorem 1 (Uniqueness). *For each $f \in L^1(0, T; L^1(\Omega))$ and $g \in L^1(\Omega)$, the problem $(1-3)$ has at most one (weak) solution $u \in C([0, T]; L^2(\Omega)) \cap C^1((0, T); L^2(\Omega))$.*

In spite of the uniqueness, the problem is still ill-posed and a regularization is necessary. For each $\epsilon > 0$, introduce the truncation mapping $P_\epsilon : L^1(\Omega) \to C^\infty(\Omega) \cap H_0^1(\Omega)$

$$P_\epsilon w(x, y) = \sum_{m,n \geq 1; m^2+n^2 \leq M_\epsilon} w_{mn} \sin(mx) \sin(ny), \quad \text{with } M_\epsilon = \frac{\ln(\epsilon^{-1})}{2T}. \tag{8}$$

In fact, P_ϵ is a finite-dimensional orthogonal projection on $L^2(\Omega)$, but it works on $L^1(\Omega)$ as well. We shall approximate the original problem by the following well-posed problem.

Theorem 2 (Well-posed problem). *For each $f \in L^1(0, T; L^1(\Omega))$ and $g \in L^1(\Omega)$, let $w \in L^1(0, T; L^2(\Omega))$ defined by*

$$w_{mn}(t) = e^{(T-t)(m^2+n^2)}(P_\epsilon g)_{mn} - \int_t^T e^{(s-t)(m^2+n^2)}(P_\epsilon f)_{mn}(s)ds, \quad \forall m, n \geq 1. \tag{9}$$

Then $w = P_\epsilon w$ and it depends continuously on (f, g), i.e. if w_i is the solution with respect to (f_i, g_i), $i = 1, 2$, then

$$\|w_1(t) - w_2(t)\|_{L^2(\Omega)} \leq \frac{2\sqrt{\ln(\varepsilon^{-1})}}{\pi\sqrt{2T}} \varepsilon^{\frac{t-T}{2T}} \left(\|g_1 - g_2\|_{L^1(\Omega)} + \|f_1 - f_2\|_{L^1(0,T;L^1(\Omega))} \right).$$

Proof. Note that $w(t)$ is well-defined because $w_{mn}(t) = 0$ if $m^2 + n^2 > M_\epsilon$. This fact also implies that $w = P_\epsilon w$. Now for two solutions w_1, w_2 we have

$$\|w_1(t) - w_2(t)\|^2_{L^2(\Omega)} = \frac{\pi^2}{4} \sum_{m,n\geq 1} |w_{1,mn}(t) - w_{2,mn}(t)|^2$$

$$= \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2 \leq M_\epsilon} \left| e^{(T-t)(m^2+n^2)}(g_1 - g_2)_{mn} - \int_t^T e^{(s-t)(m^2+n^2)}(f_1 - f_2)_{mn}(s) ds \right|^2$$

$$\leq \frac{4}{\pi^2} \sum_{m,n\geq 1; m^2+n^2 \leq M_\epsilon} \left| e^{(T-t)M_\varepsilon} \|g_1 - g_2\|_{L^1(\Omega)} + \int_t^T e^{(T-t)M_\varepsilon} \|f_1(s) - f_2(s)\|_{L^1(\Omega)} ds \right|^2$$

$$\leq \frac{4}{\pi^2} M_\varepsilon e^{2(T-t)M_\varepsilon} \left(\|g_1 - g_2\|_{L^1(\Omega)} + \|f_1 - f_2\|_{L^1(0,T;L^1(\Omega))} \right)^2 .$$

Here we have used $|v_{mn}| \leq |v|_{L^1(\Omega)}$ and the fact that

$$\#\{(m,n) \in \mathbb{Z}^2 | m,n \geq 1, m^2 + n^2 \leq M_\varepsilon\} \leq M_\varepsilon.$$

Reviewing the value of M_ε, we have the desired estimate. □

Remark 1. *A significant convenience of our method is that it is very easy to compute and represent explicitly the solution w defined by (9). Moreover, this solution is very smooth because $w(t) = P_\epsilon w(t) \in C^\infty(\Omega) \cap H^1_0(\Omega)$ for all $t \in [0,T]$.*

Remark 2. *The stability magnitude of our well-posed problem is of order $\sqrt{\ln(\varepsilon^{-1})}\varepsilon^{\frac{t-T}{2T}}$. It is much better, especially when $t = 0$, than the stability magnitudes given by quasi-reversibility method and quasi-boundary value method, for example, $\epsilon^{\frac{t-T}{T}}$ in [3, 19] and $(\epsilon \ln(\epsilon^{-1}))^{-1}$ in [4, 18].*

Our regularized solution is the solution produced directly by the well-posed problem in the previous section from the given data which works even on a very weak assumption on the exact solution.

Theorem 3 (Regularized solution)**.** *Assume that the problem* $(1-3)$ *has at most one (weak) solution $u \in C([0,T]; L^2(\Omega)) \cap C^1((0,T); L^2(\Omega))$ corresponding to $f \in L^1(0,T; L^1(\Omega))$ and $g \in L^1(\Omega)$. Let f_ϵ and g_ϵ be measured data satisfying*

$$\|f_\epsilon - f\|_{L^1(0,T;L^1(\Omega))} \leq \epsilon, \|g_\epsilon - g\|_{L^1(\Omega)} \leq \epsilon.$$

Define the regularized solution $u_\epsilon \in L^1(0,T; L^2(\Omega))$ from f_ϵ and g_ϵ as in (9). Then for each $t \in [0,T]$, $u_\epsilon(t) \in C^\infty(\Omega) \cap H^1_0(\Omega)$ and $\lim_{\varepsilon\to 0} u_\epsilon(t) = u(t)$ in $L^2(\Omega)$.

Proof. We shall use the notations P_ϵ and M_ϵ defined in (8). Note that $u_\epsilon(t) = P_\epsilon u_\epsilon(t) \in C^\infty(\Omega) \cap H_0^1(\Omega)$ as in Remark 1. Moreover using the stability in Theorem 2 we find that

$$\begin{aligned}\|u_\varepsilon(t) - u(t)\|_{L^2(\Omega)} &\leq \|P_\epsilon u_\varepsilon(t) - P_\varepsilon u(t)\|_{L^2(\Omega)} + \|P_\varepsilon u(t) - u(t)\|_{L^2(\Omega)} \\ &\leq \frac{4\sqrt{\ln(\varepsilon^{-1})}}{\pi\sqrt{2T}} \varepsilon^{\frac{T+t}{2T}} + \frac{\pi}{2}\left(\sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} |u_{mn}(t)|^2\right)^{1/2} \qquad (10)\end{aligned}$$

and it must converge to 0 as $\epsilon \to 0$. To obtain the convergence of the second term in the right-hand side of (10), we note that

$$\frac{\pi^2}{4} \sum_{m,n\geq 1} |u_{mn}(t)|^2 = \|u(t)\|^2_{L^2(\Omega)} < \infty.$$

and $M_\epsilon \to \infty$ as $\epsilon \to 0$. □

In the above theorem, we did not give an error estimate because the condition of the exact solution u is so weak (we even did not require $u(t) \in H_0^1(\Omega)$). However in practical application we may expect that the exact solution is smoother. In these cases many explicit errors estimates are available in the next section. An essential point here is that the regularized solution is the same in any case. This is a substantial pleasure for practical application because even if someones do not know how good the exact solution is they are always ensured that the regularized solution works as well as possible without any further adjustment.

3 Error estimates

From the usual viewpoint from variational method, it is natural to assume that $u(t) \in H_0^1(\Omega)$ for all $t \in [0,T]$. Moreover, if f is smooth and u is a classical solution for the heat equation (1) then $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for all $t \in [0,T]$. For these two cases we have the following explicit error estimates.

Theorem 4 (Error estimate for usual cases)**.** *Let u, u_ϵ as in Theorem 3 and let $t \in [0,T]$.*
(i) Assume that $u(t) \in H_0^1(\Omega)$. Then $\lim_{\epsilon\to 0} u_\epsilon(t) = u(t)$ in $H_0^1(\Omega)$ and

$$\|u_\varepsilon(t) - u(t)\|_{L^2(\Omega)} \leq \frac{4\sqrt{\ln(\varepsilon^{-1})}}{\pi\sqrt{2T}} \varepsilon^{\frac{T+t}{2T}} + \frac{\sqrt{2T}}{\sqrt{\ln(\varepsilon^{-1})}} \|\nabla u(t)\|_{L^2(\Omega)}.$$

(ii) Assume that $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$. Then $\lim_{\epsilon \to 0} u_\epsilon(t) = u(t)$ in $H^2(\Omega)$ and

$$\begin{aligned} \|u_\varepsilon(t) - u(t)\|_{L^2(\Omega)} &\leq \frac{4\sqrt{\ln(\varepsilon^{-1})}}{\pi\sqrt{2T}} \varepsilon^{\frac{T+t}{2T}} + \frac{2T}{\ln(\varepsilon^{-1})} \|u(t)\|_{H^2(\Omega)} \\ \|u_\varepsilon(t) - u(t)\|_{H_0^1(\Omega)} &\leq \frac{2\ln(\varepsilon^{-1})}{\pi T} \varepsilon^{\frac{T+t}{2T}} + \frac{\sqrt{2T}}{\sqrt{\ln(\varepsilon^{-1})}} \|u(t)\|_{H^2(\Omega)}^2 . \end{aligned}$$

Here we use the norm

$$\begin{aligned} \|w\|_{H_0^1}^2 &= \|\nabla w\|_{L^2}^2 = \|w_x\|_{L^2}^2 + \|w_y\|_{L^2}^2 , \\ \|w\|_{H^2}^2 &= \|w\|_{L^2}^2 + \|w\|_{H_0^1}^2 + \|w_{xx}\|_{L^2}^2 + \|w_{xy}\|_{L^2}^2 + \|w_{yx}\|_{L^2}^2 + \|w_{yy}\|_{L^2}^2 . \end{aligned}$$

Proof. (i) By using the integral by part and the Parseval equality, it is straightforward to check that if $u(t) \in H_0^1(\Omega)$ then

$$\frac{\pi^2}{4} \sum_{m,n\geq 1} (m^2+n^2)|u_{mn}(t)|^2 = \|\nabla u(t)\|_{L^2(\Omega)}^2 . \tag{11}$$

Using (11) we have

$$\sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} |u_{mn}(t)|^2 \leq \frac{1}{M_\varepsilon} \sum_{m,n\geq 1} (m^2+n^2)|u_{mn}(t)|^2 = \frac{4}{\pi^2 M_\varepsilon} \|\nabla u(t)\|_{L^2(\Omega)}^2 .$$

Substituting the latter inequality into the estimate (10) in the proof of Theorem 3, we obtain the error estimate in L^2.

To prove the convergence in H_0^1 we use the identity (11) and the stability of Theorem 2 again

$$\|\nabla u_\varepsilon(t) - \nabla u(t)\|_{L^2(\Omega)}^2 = \frac{\pi^2}{4} \sum_{m,n\geq 1} (m^2+n^2)|u_{\varepsilon,mn}(t) - u_{mn}(t)|^2$$

$$\begin{aligned} &= \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2\leq M_\varepsilon} (m^2+n^2)|u_{\varepsilon,mn}(t) - u_{mn}(t)|^2 \\ &\quad + \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} (m^2+n^2)|u_{mn}(t)|^2 \\ &\leq \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2\leq M_\varepsilon} M_\varepsilon |(P_\varepsilon u_\varepsilon)_{mn}(t) - (P_\varepsilon u)_{mn}(t)|^2 \\ &\quad + \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} (m^2+n^2)|u_{mn}(t)|^2 \\ &\leq M_\varepsilon \|P_\varepsilon u_\varepsilon(t) - P_\varepsilon u(t)\|_{L^2(\Omega)}^2 + \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} (m^2+n^2)|u_{mn}(t)|^2 \\ &\leq \frac{4(\ln(\varepsilon^{-1}))^2}{\pi^2 T^2} \varepsilon^{\frac{T+t}{T}} + \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} (m^2+n^2)|u_{mn}(t)|^2. \end{aligned} \tag{12}$$

The second term in the right-hand side in (12) converges to 0 as $\epsilon \to 0$ because the convergence in (11). Thus the convergence in H_0^1 has been proved.

(ii) We now assume that $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$. We have an identity similar to (11)

$$\begin{aligned} &\frac{\pi^2}{4} \sum_{m,n\geq 1} (m^2+n^2)^2 |u_{mn}(t)|^2 \\ = \quad & \|u_{xx}(t)\|^2_{L^2(\Omega)} + \|u_{xy}(t)\|^2_{L^2(\Omega)} + \|u_{yx}(t)\|^2_{L^2(\Omega)} + \|u_{yy}(t)\|^2_{L^2(\Omega)} . \end{aligned} \tag{13}$$

The error estimate in $L^2(\Omega)$ follows (10) and the following inequality

$$\sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} |u_{mn}(t)|^2 \leq \frac{1}{M_\varepsilon^2} \sum_{m,n\geq 1} (m^2+n^2)^2 |u_{mn}(t)|^2 \leq \frac{4}{\pi^2 M_\varepsilon^2} \|u(t)\|^2_{H^2(\Omega)} .$$

Similarly, from (12) and the estimate

$$\begin{aligned} \sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} (m^2+n^2) |u_{mn}(t)|^2 \quad &\leq \quad \frac{1}{M_\varepsilon} \sum_{m,n\geq 1} (m^2+n^2)^2 |u_{mn}(t)|^2 \\ &\leq \quad \frac{4}{\pi^2 M_\varepsilon} \|u(t)\|^2_{H^2(\Omega)} , \end{aligned}$$

we find that

$$\|\nabla u_\varepsilon(t) - \nabla u(t)\|^2_{L^2(\Omega)} \leq \frac{\pi^2 (\ln(\varepsilon^{-1}))^2}{4T^2} \varepsilon^{\frac{T+t}{T}} + \frac{1}{M_\varepsilon} \|u(t)\|^2_{H^2(\Omega)} .$$

Using the inequality $a + b \leq (\sqrt{a} + \sqrt{b})^2$ we obtain the error estimate in H_0^1.

Finally we prove the convergence in $H^2(\Omega)$. Similarly to (12) we have

$$\begin{aligned} & \|(u_\varepsilon - u)_{xx}(t)\|^2_{L^2} + \|(u_\varepsilon - u)_{xy}(t)\|^2_{L^2} + \|(u_\varepsilon - u)_{yx}(t)\|^2_{L^2} + \|(u_\varepsilon - u)_{yy}(t)\|^2_{L^2} \\ = \quad & \frac{\pi^2}{4} \sum_{m,n\geq 1} (m^2+n^2)^2 |u_{\varepsilon,mn}(t) - u_{mn}(t)|^2 \\ \leq \quad & \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2\leq M_\varepsilon} M_\varepsilon^2 |u_{\varepsilon,mn}(t) - u_{mn}(t)|^2 \\ & \quad + \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} (m^2+n^2)^2 |u_{mn}(t)|^2 \\ \leq \quad & M_\varepsilon^2 \|P_\varepsilon u_\varepsilon(t) - P_\varepsilon u(t)\|^2_{L^2(\Omega)} + \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} (m^2+n^2)^2 |u_{mn}(t)|^2 \\ \leq \quad & \frac{2(\ln(\varepsilon^{-1}))^3}{\pi^2 T^3} \varepsilon^{\frac{T+t}{T}} + \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} (m^2+n^2)^2 |u_{mn}(t)|^2 \to 0 \end{aligned} \tag{14}$$

as $\epsilon \to 0$ due to the convergence in (13). $\square$

Remark 3. *In Theorem 4 we have pointwise estimates due to the pointwise condition on the exact solution u. As a consequence, we shall immediately obtain a uniform convergence whenever the corresponding uniform condition is imposed. For example, if the exact solution u is in $C([0,T];H_0^1(\Omega))$ or $C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$ then we have the estimates $\|u_\varepsilon - u\|_{C([0,T];L^2(\Omega))}$ and $\|u_\varepsilon - u\|_{C([0,T];H_0^1(\Omega))}$, respectively, in the same form of estimates in Theorem 4.*

Remark 4. *The error estimates in Theorem 4 work well no matter $t>0$ or $t=0$. In many earlier works, we find that the error $\|u_\epsilon(0)-u(0)\|_{L^2(\Omega)}$ is often not given (e.g. [17]) and an explicit error estimate in $H_0^1(\Omega)$ is not available (e.g. [3, 7, 4, 17, 19, 18]).*

In Theorem 4 (ii), an error estimate in $H^2(\Omega)$ is not given because we do not have enough information on the exact solution (we just know $u(t)\in H^2(\Omega)\cap H_0^1(\Omega)$). However, when u is smoother then an explicit estimate in $H^2(\Omega)$ may be derived. In the last theorem, we shall give the error estimates in some special cases when the exact solution is very good. We see from the proof of Theorem 4 that the facts $u(t)\in H_0^1(\Omega)$ and $u(t)\in H^2(\Omega)\cap H_0^1(\Omega)$ are equivalent to

$$\sum_{m,n\geq 1}(m^2+n^2)^k|u_{mn}(t)|^2<\infty$$

with $k=1,2$, respectively. We shall see that from the latter condition with $k>2$ we may improve the estimate, and in particular give an error estimate in $H^2(\Omega)$. We next consider a stronger condition similar to (in fact, weaker than)

$$\sup_{t\in[0,T]}\sum_{m,n\geq 1}e^{2T(m^2+n^2)}|u_{mn}(t)|^2<\infty \tag{15}$$

which is a two-dimensional version of the condition (4) in [14]. Such a condition seems essential to solve the nonlinear problem. Although it is quite strict for the linear case, as we discussed in the first section, if the above condition (15) holds then we have a very good convergence rate which is of order $(\ln(\epsilon^{-1}))^{3/2}\sqrt{\epsilon}$.

Theorem 5 (Error estimate for special cases)**.** *Let u, u_ϵ as in Theorem 3 and let $t\in[0,T]$. (i) Assume that*

$$E_k(t)=\sum_{n,m=1}^{\infty}(n^2+m^2)^k u_{mn}^2(t)<\infty \tag{16}$$

for some constant $k > 2$. Then

$$\begin{aligned}
\|u_\varepsilon(t) - u(t)\|_{L^2(\Omega)} &\leq \frac{4\sqrt{\ln(\varepsilon^{-1})}}{\pi\sqrt{2T}}\varepsilon^{\frac{T+t}{2T}} + \frac{\pi\sqrt{E_k(t)}}{2}\left(\frac{2T}{\ln(\varepsilon^{-1})}\right)^{\frac{k}{2}} \\
\|u_\varepsilon(t) - u(t)\|_{H_0^1(\Omega)} &\leq \frac{2\ln(\varepsilon^{-1})}{\pi T}\varepsilon^{\frac{T+t}{2T}} + \frac{\pi\sqrt{E_k(t)}}{2}\left(\frac{2T}{\ln(\varepsilon^{-1})}\right)^{\frac{k-1}{2}} \\
\|u_\varepsilon(t) - u(t)\|_{H^2(\Omega)} &\leq \frac{2(\ln(\varepsilon^{-1}))^{3/2}}{\pi T\sqrt{T}}\varepsilon^{\frac{T+t}{2T}} + \frac{3\pi\sqrt{E_k(t)}}{2}\left(\frac{2T}{\ln(\varepsilon^{-1})}\right)^{\frac{k-2}{2}}.
\end{aligned}$$

Here we assume $\epsilon \leq e^{-2T}$ for the estimate in $H^2(\Omega)$.

(ii) Assume that

$$F_r(t) = \sum_{m,n\geq 1} e^{2r(m^2+n^2)}|u_{mn}(t)|^2 < \infty$$

for some constant $r > 0$. Then

$$\begin{aligned}
\|u_\varepsilon(t) - u(t)\|_{L^2(\Omega)} &\leq \frac{4\sqrt{\ln(\varepsilon^{-1})}}{\pi\sqrt{2T}}\varepsilon^{\frac{T+t}{2T}} + \frac{\pi\sqrt{F_r(t)}}{2}\varepsilon^{\frac{r}{2T}} \\
\|u_\varepsilon(t) - u(t)\|_{H_0^1(\Omega)} &\leq \frac{2\ln(\varepsilon^{-1})}{\pi T}\varepsilon^{\frac{T+t}{2T}} + \frac{\pi\sqrt{F_r(t)\ln(\varepsilon^{-1})}}{2\sqrt{2T}}\varepsilon^{\frac{r}{2T}} \\
\|u_\varepsilon(t) - u(t)\|_{H^2(\Omega)} &\leq \frac{6(\ln(\varepsilon^{-1}))^{3/2}}{\pi T\sqrt{T}}\varepsilon^{\frac{T+t}{2T}} + \frac{3\pi\sqrt{F_r(t)}\ln(\varepsilon^{-1})}{4T}\varepsilon^{\frac{r}{2T}}.
\end{aligned}$$

Here we assume $\epsilon \leq e^{-2T}$ for the estimate in $H_0^1(\Omega)$, and $\epsilon \leq e^{-4T}$ for the estimate in $H^2(\Omega)$.

Proof. (i) We use the same way of the proof of Theorem 4. We shall prove the error estimates in $H^2(\Omega)$ (the other ones are similar and easier). From

$$\sum_{m,n\geq 1; m^2+n^2 > M_\varepsilon} (m^2+n^2)^2|u_{mn}(t)|^2 \leq \frac{1}{M_\varepsilon^{k-2}}\sum_{m,n\geq 1}(m^2+n^2)^k|u_{mn}(t)|^2 \leq \frac{E_k(t)}{M_\varepsilon^{k-2}}$$

and (14) we find that

$$\begin{aligned}
&\|(u_\varepsilon - u)_{xx}(t)\|_{L^2}^2 + \|(u_\varepsilon - u)_{xy}(t)\|_{L^2}^2 + \|(u_\varepsilon - u)_{yx}(t)\|_{L^2}^2 + \|(u_\varepsilon - u)_{yy}(t)\|_{L^2}^2 \\
&\leq \frac{2(\ln(\varepsilon^{-1}))^3}{\pi^2 T^3}\varepsilon^{\frac{T+t}{T}} + \frac{\pi^2 E_k}{4M_\varepsilon^{k-2}} \leq \left(\frac{4}{\pi}M_\varepsilon^{3/2}\varepsilon^{\frac{T+t}{2T}} + \frac{\pi\sqrt{E_k(t)}}{2}M_\varepsilon^{-\frac{k-2}{2}}\right)^2.
\end{aligned}$$

Using

$$\|w\|_{H^2} \leq \|w\|_{L^2} + \|w\|_{H_0^1} + \sqrt{\|w_{xx}\|_{L^2}^2 + \|w_{xy}\|_{L^2}^2 + \|w_{yx}\|_{L^2}^2 + \|w_{yy}\|_{L^2}^2} \tag{17}$$

and $M_\epsilon \geq 1$ we conclude the desired estimate in $H^2(\Omega)$.

(ii) From (10) and

$$\sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} |u_{mn}(t)|^2 \leq e^{-2rM_\varepsilon} \sum_{m,n\geq 1} e^{2r(m^2+n^2)}|u_{mn}(t)|^2 \leq F_r(t)\varepsilon^{\frac{r}{T}}$$

we get the error estimate in $L^2(\Omega)$.

Note that the function $\xi \mapsto e^\xi/\xi$ is increasing when $\xi \geq 1$. Thus

$$(m^2+n^2) \leq M_\varepsilon e^{2r(m^2+n^2-M_\varepsilon)} \text{ when } m^2+n^2 > M_\varepsilon \geq 1.$$

It implies that

$$\sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} (m^2+n^2)|u_{mn}(t)|^2 \leq M_\varepsilon \sum_{m,n\geq 1} e^{2r(m^2+n^2-M_\varepsilon)}|u_{mn}(t)|^2 \leq M_\varepsilon F_r(t)\varepsilon^{\frac{r}{\cdot}T}$$

The error estimate in $H_0^1(\Omega)$ follows the above estimate and (12).

Similarly, because the function $\xi \mapsto e^\xi/\xi^2$ is increasing when $\xi \geq 2$, we find that

$$(m^2+n^2)^2 \leq M_\varepsilon^2 e^{2r(m^2+n^2-M_\varepsilon)} \text{ if } m^2+n^2 > M_\varepsilon \geq 2.$$

It follows that

$$\sum_{m,n\geq 1; m^2+n^2>M_\varepsilon} (m^2+n^2)|u_{mn}(t)|^2 \leq M_\varepsilon^2 \sum_{m,n\geq 1} e^{2r(m^2+n^2-M_\varepsilon)}|u_{mn}(t)|^2 \leq M_\varepsilon^2 F_r(t)\varepsilon^{\frac{r}{T}}.$$

Thus (14) reduces to

$$\begin{aligned}
&\|(u_\varepsilon - u)_{xx}(t)\|_{L^2}^2 + \|(u_\varepsilon - u)_{xy}(t)\|_{L^2}^2 + \|(u_\varepsilon - u)_{yx}(t)\|_{L^2}^2 + \|(u_\varepsilon - u)_{yy}(t)\|_{L^2}^2 \\
\leq\ & \frac{2(\ln(\varepsilon^{-1}))^3}{\pi^2 T^3}\varepsilon^{\frac{T+t}{T}} + M_\varepsilon^2 F_r(t)\varepsilon^{\frac{r}{T}} \leq \left(\frac{4}{\pi}M_\varepsilon^{3/2}\varepsilon^{\frac{T+t}{2T}} + \frac{\pi\sqrt{F_r(t)}}{2}M_\varepsilon \varepsilon^{\frac{r}{2T}}\right)^2.
\end{aligned}$$

Using (17) again and $M_\epsilon \geq 1$ we conclude the error estimate in $H^2(\Omega)$. □

Remark 5. *If (15) holds, i.e.* $\sum_{m,n\geq 1} e^{2T(m^2+n^2)}|u_{mn}(t)|^2 < \infty$*, then applying Theorem 5 in the case* $r = T$ *we get*

$$\begin{aligned}
\sup_{t\in[0,T]} \|u_\varepsilon(t) - u(t)\|_{L^2(\Omega)} &\leq C\sqrt{\varepsilon}, \\
\sup_{t\in[0,T]} \|u_\varepsilon(t) - u(t)\|_{H_0^1(\Omega)} &\leq C\ln(\varepsilon^{-1})\sqrt{\varepsilon}, \\
\sup_{t\in[0,T]} \|u_\varepsilon(t) - u(t)\|_{H^2(\Omega)} &\leq C(\ln(\varepsilon^{-1}))^{3/2}\sqrt{\varepsilon}.
\end{aligned}$$

Notice that in [11], under a similar condition, Liu gave the error estimate (See Theorem 3.3, page 466)

$$\|g_\alpha^\epsilon - g_0\|_{L^2(\Omega)} \le C(\epsilon^{\frac{1}{2}})^{1-\frac{t_0}{T}}$$

Let $t_0 = 0$, we get

$$\|g_\alpha^\epsilon - g_0\|_{L^2(\Omega)} \le C\sqrt{\epsilon^{\frac{1}{2}}}$$

Thus at $t = 0$, our method gave the same order error in L^2-norm as the method of Liu [11]. However, the strong point of our paper is that the error estimates in $H_0^1(\Omega)$ or $H^2(\Omega)$ established, and also of Hölder type (in fact, they are better than any estimate of order ϵ^q with $q \in (0, 1/2)$). They are not given in [11].

Remark 6. *The truncated regularization method is a very simple and effective method for solving some ill-posed problems and it has been successfully applied to some inverse heat conduction problems [5, 8, 13]. Recently, in [14] many applications for a model of the Helmholtz equation are introduced and a Fourier method was applied for solving a Cauchy problem for the Helmholtz equation. In [9], ChuLiFu and his group used the truncated method to solve the backward heat in the unbounded region and established the logarithimic order of the form*

$$\begin{aligned}&\|u(.,t) - u_{\delta,\xi_{\max}}\| \\ &\quad\le E^{1-\frac{t}{T}} \left(\ln \frac{E}{\delta}\right)^{\frac{(t-T)s}{2T}} \left(1 + \left(\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta} + \ln(\ln\frac{E}{\delta})^{\frac{-s}{2T}}}\right)^{\frac{s}{2}}\right). \end{aligned} \quad (18)$$

And in [20], the authors gave the following estimates

$$\begin{aligned}&\|w^{\beta,a_\beta}(.,.,t) - u(.,.,t)\| \\ &\le \beta^{t/T}(\ln\frac{1}{\beta})^{\frac{-\alpha(T-t)}{2T}}\Big(\exp(k^2(T-t)^2) + Q(\beta,t,u)(\ln\frac{1}{\beta})^{\frac{\alpha}{2}}\Big).\end{aligned} \quad (19)$$

And in [21], Trong and Tuan only established the logarithmic form as follows

$$\|u(.,.,t) - u^\epsilon(.,.,t)\| \le \frac{C}{1 + \ln(\frac{T}{\epsilon})} \quad (20)$$

Note that the errors (18), (19) and (20) are the same order as Theorem 5 (i). However,the logarithmic type estimate is, in general, much worse than any Hölder type estimate, i.e. ϵ^q for some $q > 0$. In Theorem 5 (ii) we also establish this type of estimates, which are not given in [9, 20, 21]. It worth mentioning that our regularized solution is unique, in all cases. This proves that our method is effective.

Remark 7. *Sometime, it is also important to considerd the 2-D backward heat for a general two-dimensional domain, e.g. [12]. In this case to apply the truncation method, we need to consider the spectral problem of operator* $-\Delta$ *in this domain (with homogeneous Dirichlet boundary condition). However, this question is not always solvable explicitly and this is a disadvantage point of our method.*

4 Numerical experiments

In this section we give some numerical experiments for our method. For simplicity, we shall recover the initial temperature at $t = 0$ from the final data at $T = 1$.
Example 1. Consider the problem

$$u_t - \Delta u = f(x, y, t) \equiv 3e^t \sin(x)\sin(y),$$

with the final condition

$$u(x, y, 1) = g(x, y) \equiv e \sin(x)\sin(y).$$

Problem (1)-(3) with exact data (f, g) has the exact solution

$$u(x, y, t) = e^t \sin(x)\sin(y).$$

For any $n = 1, 2, ...$, let us take the measured data

$$f_n = f, g_n(x, y) = g(x, y) + n \sin(nx)\sin(ny).$$

Then Problem (1)-(3) with measured data (f_ϵ, g_ϵ) has corresponding solution

$$\widetilde{u}_n(x, y, t) = u(x, y, t) + \frac{1}{n} e^{n^2(1-t)} \sin(nx)\sin(ny).$$

We see that

$$\begin{aligned} \|g_n(x, y) - g(x, y)\|_{L^1(\Omega)} &= \frac{4}{n} \to 0, \\ \|\widetilde{u}_n(.,.,0) - u(.,.,0)\|_{L^2(\Omega)} &= \frac{e^{n^2}}{n} \to +\infty \end{aligned}$$

It means that if n is large then a small error of data might cause a large error of solutions. Therefore, the problem is really unstable and hence a regularization is necessary. Using the regularization of Theorem 2 correspoding $\varepsilon = 4/n$, we see that when $n > 4e^4$ then the regularized solution at $t = 0$ is

$$u_\varepsilon(x, y) = \sin(x)\sin(y),$$

which coincides the exact solution $u(., ., 0)$. In this example, our method works very well because the exact solution's form is of a truncated Fourier series.

Example 2. Consider the problem

$$u_t - \Delta u = f(x, y, t) \equiv \sin(x)\left(ty^3 - \pi t y^2 - 6ty + 6y + 2t\pi - 2\pi\right)$$

with the final condition

$$u(x, y, 1) = g(x, y) \equiv 0$$

The exact solution of the latter equation is

$$u(x, y, t) = (1 - t)\sin(x)y^2(\pi - y).$$

For any $n = 1, 2, ...$, take the measured data

$$f_n = f, g_n(x, y) = \frac{1}{4n}\sin(nx)\sin(ny).$$

Then the disturbed solution is

$$\widetilde{u}_n(x, y, t) = u(x, y, t) + \frac{1}{4n}e^{n^2(1-t)}\sin(nx)\sin(ny).$$

We see that

$$\begin{aligned} \|g_n(x, y) - g(x, y)\|_{L^1(\Omega)} &= \frac{1}{n} \to 0, \\ \|\widetilde{u}_n(., ., 0) - u(., ., 0)\|_{L^2(\Omega)} &= \frac{e^{n^2}}{4n} \to +\infty. \end{aligned}$$

Thus the problem in this case is also unstable. We now compute the regularized solutions by using the regularization method introduced in the previous sections with $\varepsilon = 1/n$. The effect of our regularization is represented via Table 1 below, where we denote by $u_\epsilon := u_\epsilon(., ., 0)$ the regularized value at $t = 0$, $\widetilde{u}_\epsilon := \widetilde{u}_n(., ., 0)$ the disturbed value (with $\epsilon = 1/n$), and $u_0 := u(., ., 0)$ the exact value. We can see that while the errors between the disturbed solution and the exact solution is extremely large, the error between the regularized solution and the exact solution is acceptable, even in H_0^1-norm.

$\varepsilon = \frac{1}{n}$	u_ε	$\|\widetilde{u}_\epsilon - u_0\|_{L^2}$	$\|u_\varepsilon - u_0\|_{L^2}$	$\|u_\varepsilon - u_0\|_{H_0^1}$
10^{-3}	$4\sin(x)\sin(y)$	exp(999992)	2.388527	5.506293
10^{-5}	$4\sin(x)\sin(y) - \frac{3}{2}\sin(x)\sin(2y)$	$\exp(10^{10})$	0.391677	1.600311
10^{-9}	$4\sin(x)\sin(y) - \frac{3}{2}\sin(x)\sin(2y)$ $+ \frac{4}{27}\sin(x)\sin(3y)$	$\exp(10^{18})$	0.315050	1.421075
10^{-15}	$4\sin(x)\sin(y) - \frac{3}{2}\sin(x)\sin(2y)$ $+ \frac{4}{27}\sin(x)\sin(3y)$ $-\frac{3}{16}\sin(x)\sin(4y)$	$\exp(10^{30})$	0.111858	0.738103

Table 1. Errors between disturbed solution, regularized solution and exact solution.

Acknowledgments

Most part of the work was done when the first author was a student of University of Science, Vietnam National University at HoChiMinh City. The second and the third authors are supported by the Council for Natural Sciences of Vietnam. We thank the referees for constructive comments leading to the improved version of the paper.

References

[1] S. Agmon, L. Nirenberg, Properties of solutions of ordinary differential equations in Banach spaces, *Comm. Pure. Appl. Math.* **16** (1963), p. 121-139.

[2] F. Black, M. Scholes, The Pricing of Options and Corporate Liabilities, *J. Political Economy* **81**(3) (1973).

[3] G. W. Clark and S. F. Oppenheimer, Quasireversibility methods for non-well posed problems, *Elect. J. Diff. Eqns.* **8** (1994), p. 1-9.

[4] M. Denche and K. Bessila, A modified quasi-boundary value method for ill-posed problems, *J. Math. Anal. Appl.* **301** (2005), p. 419-426.

[5] L. Elden, F. Berntsson; T. Reginska, Wavelet and Fourier methods for solving the sideways heat equation. *SIAM J. Sci. Comput.* **21**(6) (2000), p. 2187-2205.

[6] L.C. Evans, *Partial Differential Equations*, American Mathematical Society (1998).

[7] R.E. Ewing, The approximation of certain parabolic equations backward in time by Sobolev equations, *SIAM J. Math. Anal.* **6** (1975), p. 283-294.

[8] C.L. Fu, Simplified Tikhonov and Fourier regularization methods on a general sideways parabolic equation, *J. Comput. Appl. Math.* **167** (2004), p. 449-463.

[9] C.L. Fu, X.T Xiong, Z. Qian, Fourier regularization for a backward heat equation. *J. Math. Anal. Appl.* **331**(1) (2007), p. 472-480.

[10] R. Lattès, J.L. Lions, *Méthode de Quasi-réversibilité et Applications*, Dunod, Paris (1967).

[11] J.J. Liu, Numerical solution of forward and backward problem for 2-D heat equation, *J. Comput.Appl. Math.*, **145** (2002), p. 459-482.

[12] J. Cheng, J.J. Liu, A quasi Tikhonov regularization for a two-dimensional backward heat problem by a fundamental solution, *Inverse Problems* **24** 065012 (2008).

[13] Z. Qian, C.L. Fu, Regularization strategies for a two-dimensional inverse heat conduction problem, *Inverse Problems* **23** 1053 (2007).

[14] T. Regińska, K. Regiński, Approximate solution of a Cauchy problem for the Helmholtz equation, *Inverse Problems* **22** (2006), p. 975-989.

[15] R.E. Showalter, Cauchy problem for hyper-parabolic partial differential equations, *Trends in the Theory and Practice of Non-Linear Analysis*, Elsevier (1983).

[16] A.N. Tikhonov, V.Y. Arsenin, *Solutions of Ill-posed Problems*, Winston, Washington (1977).

[17] D.D. Trong and N.H. Tuan, Regularization and error estimates for nonhomogeneous backward heat problems, *Electron. J. Diff. Eqns.* **04** (2006), p. 1-10.

[18] D.D. Trong and N.H. Tuan, A nonhomogeneous backward heat problem: Regularization and error estimates , *Electron. J. Diff. Eqns.* **33** (2008), p. 1-14.

[19] D.D. Trong, P.H. Quan,T.V. Khanh, N.H. Tuan, A nonlinear case of the 1-D backward heat problem: Regularization and error estimate, *Zeitschrift Analysis und ihre Anwendungen*, **26**(2)(2007), p. 231-245.

[20] D.D. Trong , N.H. Tuan, Remarks on a 2-D nonlinear backward heat problem using a truncated Fourier series method, *Electron. J. Diff. Eqns.* **77**(2009), p. 1-13.

[21] D.D. Trong, N.H. Tuan, A new regularized method for two dimensional non-homogeneous backward heat problem, *Appl. Math. Comput.* **215** (3) (2009), p. 873-880.