Prerequisites

P.1. Measures & integration [Literature: Halmos, Bauer]

Banach-Tarski paradox $\Rightarrow$ σ-additive set functions cannot be defined on all subsets of a given set

$\Rightarrow$ define measure only on "good" subsets (i.e. σ-algebra)

P.1. Definition $M \neq \emptyset$ a set.

- σ-algebra (or: σ-field or: measurable sets)

  $\mathcal{A} \subseteq \mathcal{P}(M)$ (power set of M) s.t.

  * $\emptyset, M \in \mathcal{A}$
  * $B \in \mathcal{A} \Rightarrow B^c \in \mathcal{A}$ (complement)
  * $B_j \in \mathcal{A}\ \forall j \in \mathbb{N} \Rightarrow \bigcup_{j\in\mathbb{N}} B_j \in \mathcal{A}$

- measure

  $\mu : \mathcal{A} \to [0,\infty]$ (σ-algebra) s.t.

  * $\mu(\emptyset) = 0$
  * σ-additivity $\rightsquigarrow$ $B_j \in \mathcal{A}\ \forall j \in \mathbb{N}$ and $B_j \cap B_k = \emptyset\ \forall j \neq k \in \mathbb{N}$

    $\Rightarrow \mu\left(\bigcup_{j\in\mathbb{N}} B_j\right) = \sum_{j\in\mathbb{N}} \mu(B_j)$

- μ-null set : $N \in \mathcal{A}$ with $\mu(N) = 0$

P.2. Examples

- Borel-σ-algebra $\mathcal{B}(\mathbb{R}^d)$ in $\mathbb{R}^d$:

  smallest σ-algebra containing all open sets in $\mathbb{R}^d$

  * $\mathcal{B}(\mathbb{R}^d) \neq \mathcal{P}(\mathbb{R}^d)$ but non-Borel sets are very exotic
  * $\mathcal{B}(\mathbb{R}^d)$ contains all closed sets in $\mathbb{R}^d$
  * analogous: $\mathcal{B}(\mathbb{C})$

- Dirac measure on $\mathbb{R}^d$ (concentrated at $x_0 \in \mathbb{R}^d$)

$$\delta_{x_0} : \mathcal{B}(\mathbb{R}^d) \to \{0,1\}$$
$$B \mapsto \delta_{x_0}(B) := \mathbb{1}_B(x_0) := \begin{cases} 1, & x_0 \in B \\ 0, & x_0 \notin B \end{cases}$$

indicator function of set B

- Lebesgue (-Borel) measure on $\mathbb{R}^d$

Theorem. $\exists_1$ unique measure $\lambda^d : \mathcal{B}(\mathbb{R}^d) \to [0,\infty]$ s.t.

$$\lambda^d\Big(\bigtimes_{j=1}^{d} [a_j, b_j[\Big) = \prod_{j=1}^{d} (b_j - a_j)$$

rectangle in $\mathbb{R}^d$

P.3. Definition $\mathcal{A}$ σ-algebra on M, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $f : M \to \mathbb{K}$

- f elementary fct. $:\Leftrightarrow \begin{cases} \exists J \in \mathbb{N}\ \exists \alpha_j \geq 0\ \exists A_j \in \mathcal{A},\ j = 1,\dots,J, \\ A_j \cap A_k = \emptyset\ \forall j \neq k : \ f = \sum_{j=1}^{J} \alpha_j \mathbb{1}_{A_j} \end{cases}$

- f ($\mathcal{A}$-) measurable $:\Leftrightarrow \forall B \in \mathcal{B}(\mathbb{K}) : f^{-1}(B) = \{x \in M : f(x) \in B\} \in \mathcal{A}$

  * pre-images of measurable sets are measurable
  * non-measurable functions very exotic!
  * elementary functions are measurable
  * Lemma. $f \geq 0$ measurable $\Rightarrow$
    $\exists (f_n)_{n\in\mathbb{N}}$ sequ. of elementary fct.'s with $f_{n+1} \geq f_n$ and $\lim_{n\to\infty} f_n = f$

- (μ-) Integral μ a measure on $\mathcal{A}$

  * f elementary fct.: $\displaystyle\int_M d\mu(x)\, f(x) := \sum_{j=1}^{J} \alpha_j\, \mu(A_j)$
  * $f \geq 0$ measurable:
  $$\int_M d\mu(x)\, f(x) := \lim_{n\to\infty} \int_M d\mu(x)\, f_n(x)$$
  element. fct. from above Lemma

  (value $+\infty$ allowed!)

  * $f : M \to \mathbb{R}$ measurable:

  let $f_\pm := \max\{\pm f, 0\}$ (positive/negative part) $\Rightarrow$
  * $f = f_+ - f_-$
  * $0 \leq f_\pm$ measurable

f (μ-)integrable $:\Leftrightarrow$ f measurable and $\int_M d\mu(x)\, f_\pm(x) < \infty$

in this case: $\int_M d\mu(x)\, f(x) := \int_M d\mu(x)\, f_+(x) - \int_M d\mu(x)\, f_-(x)$

(integrability cond$^{\underline{n}}$ avoids "$\infty - \infty$")

- $f : M \to \mathbb{C}$ (μ-)integrable $:\Leftrightarrow$ $\mathrm{Re} f$, $\mathrm{Im} f$ integrable

  in this case: $\int_M d\mu(x)\, f(x) = \int_M d\mu(x)\, (\mathrm{Re} f)(x) + i \int_M d\mu(x)\, (\mathrm{Im} f)(x)$

$\boxed{\text{P.4. Theorem}}$ μ measure on $\mathcal{A}$, $f, g : M \to \mathbb{C}$ measurable. Then

(a) f integrable $\Leftrightarrow$ $|f|$ integrable

(b) f integrable and $f = g$ (μ-)a.e. ($:\Leftrightarrow$ $\exists$ μ-null set $N \subset M$: $f(x) = g(x)$ $\forall x \in M \setminus N$)

$\Rightarrow$ $\int_M d\mu(x)\, f(x) = \int d\mu(x)\, g(x)$

(c) $f \geq 0$ μ-a.e. and $\int_M d\mu(x)\, f(x) = 0$ $\Rightarrow$ $f = 0$ μ-a.e.

Next: the 2 basic results for interchanging integrat$^{\underline{n}}$ and limits:

$\boxed{\text{P.5. Theorem}}$ $\forall n \in \mathbb{N}$ let $f_n : M \to \mathbb{C}$ measurable

(a) Monotone convergence

If $0 \leq f_n \leq f_{n+1}$ $\forall n \in \mathbb{N}$, then $f := \lim_{n\to\infty} f_n$ measurable

and $\lim_{n\to\infty} \int_M d\mu(x)\, f_n(x) = \int_M d\mu(x)\, f(x)$ (possibly $+\infty$)

(b) Dominated convergence

If $\exists h : M \to [0, \infty]$ s.t. $|f_n| \leq h$ μ-a.e., $\int_M d\mu(x)\, h(x) < \infty$

and $\exists f : M \to \mathbb{C}$ measurable s.t. $f = \lim_{n\to\infty} f_n$ μ-a.e.,

then

$$\lim_{n\to\infty} \int_M d\mu(x)\, f_n(x) = \int_M d\mu(x)\, f(x)$$

P.6. Definition $\mathcal{A}$ σ-algebra on M; $\mu, \nu : \mathcal{A} \to [0,\infty]$ measures

(a) μ <u>σ-finite</u> $:\Leftrightarrow$ $\exists\, M_j \in \mathcal{A}, j \in \mathbb{N}$: $\bigcup_{j\in\mathbb{N}} M_j = M$ and
$\mu(M_j) < \infty \quad \forall j \in \mathbb{N}$.

(b) μ has a <u>density</u> h w.r.t. ν (or Radon-Nikodym derivative) in symbols $h = \frac{d\mu}{d\nu}$ $\Bigg\} :\Leftrightarrow \Bigg\{$ $\exists h : M \to [0,\infty[$: $\mu(B) = \int_B d\nu(x)\, h(x)$ $\forall B \in \mathcal{A}$

(c) μ is <u>absolutely continuous</u> (<u>w.r.t.</u> ν), in symbols $\mu \ll \nu$ $\Bigg\} :\Leftrightarrow \Bigg\{$ $\nu(B) = 0$ for $B \in \mathcal{A}$ $\Rightarrow \mu(B) = 0$

(d) μ and ν are <u>singular</u>, in symbols $\mu \perp \nu$ $\Bigg\} :\Leftrightarrow \Bigg\{$ $\exists\, M_0 \in \mathcal{A}$: $\mu(M_0) = 0$ and $\nu(M_0^c) = 0$

<u>P.7. Examples</u>

(a) λ^d σ-finite on $\mathbb{R}^d$ $\quad \left(\mathbb{R}^d = \bigcup_{n\in\mathbb{N}} [-n,n]^d \right)$

(b) $\lambda^d \perp \delta_{x_0} \quad \forall x_0 \in \mathbb{R}^d$

P.8. Theorem (<u>Radon-Nikodym</u>) Let μ, ν be σ-finite measures on $\mathcal{A}$

Then $\exists_1$ measure μ_{ac} $\exists_1$ measure μ_s s.t. $\mu = \mu_{ac} + \mu_s$ ← "singular"
with $\mu_{ac} \ll \nu$ and $\mu_s \perp \nu$ ($\Rightarrow \mu_{ac} \perp \mu_s$). Moreover μ_{ac} has a density w.r.t. ν.

For special case $\mu_s = 0$

P.9. Corollary μ, ν as above. Then

$$\mu \ll \nu \iff \mu \text{ has a density w.r.t. } \nu$$

$\boxed{\text{P.10. Corollary}}$ (Lebesgue decomposition) Let μ be a σ-finite Borel measure on $\mathbb{R}$ (i.e. on $\mathcal{B}(\mathbb{R})$)

Then $\exists$ unique Borel measures $\mu_{ac}, \mu_{sc}, \mu_{pp}$ on $\mathbb{R}$ s.t.

- $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$ — singular continuous, pure point
- $\mu_{ac} \ll \lambda$
- μ_{pp} is purely atomic ($\Leftrightarrow$: pure point), i.e. $\exists\, x_n \in \mathbb{R}, m_n \in]0,\infty[$ s.t.
$$\mu_{pp} = \sum_{n\in\mathbb{N}} m_n \delta_{x_n}$$
- $\mu_{sc} \perp \lambda$ and $\mu_{sc}(\{x\}) = 0 \quad \forall x \in \mathbb{R}$

In particular $\mu_{sc} \perp \mu_{pp}$, $\mu_{pp} \perp \lambda$.

Proof. Define $\mu_{pp} := \sum_{x\in\mathbb{R}} \mu_s(\{x\})\, \delta_x \quad \Rightarrow \mu_{pp}(B) \leq \mu_s(B) \leq \mu(B)$
$\forall B \in \mathcal{B}(\mathbb{R})$

$\Rightarrow \mu_{pp}$ σ-finite $\Rightarrow M = \bigcup_{j\in\mathbb{N}} M_j$ with $\mu_{pp}(M_j) = \sum_{x\in M_j} \mu_s(\{x\}) < \infty$

$\Rightarrow \exists$ at most countably many $x_n \in M_j$ with $\mu_s(\{x_n\}) > 0$
$\Rightarrow$ " " " " " $x_n \in M$ " "

Define $\mu_{sc}(B) := \mu_s(B) - \mu_{pp}(B) \quad \forall B \in \mathcal{B}(\mathbb{R}) \Rightarrow$ claim ∎

P.11 Remark

(a) With $\mu_c := \mu_{ac} + \mu_{sc}$ (continuous part) we get
$$\mu = \mu_c + \mu_{pp} = \mu_{ac} + \mu_s$$

(b) μ_{sc} supported on Lebesgue null set, but without atoms

P.2. Banach and Hilbert spaces

[Literature: Reed/Simon vol. 1, Conway, Lax]

P.12. Definition X a vector space (over $\mathbb{C}$) with norm $\|\cdot\|$

- X Banach $:\Leftrightarrow$ X complete (i.e. every Cauchy sequ. has limit in X)
- X separable $:\Leftrightarrow$ $\exists$ countable dense set in X
- (topol.) dual space $X^* := \left\{ \ell : X \to \mathbb{C} \text{ linear and } \|\ell\|_* := \sup_{0 \neq x \in X} \frac{|\ell(x)|}{\|x\|} < \infty \right\}$

  [X^* is always Banach space]
- Notions of convergence in X for $(x_n)_n \subset X$, $x \in X$
  - (norm or strong) convergence: $\lim_{n\to\infty} \|x_n - x\| = 0$

    in symbols $x_n \xrightarrow{n\to\infty} x$
  - weak convergence: $\forall \ell \in X^*: \lim_{n\to\infty} \big(\ell(x_n) - \ell(x)\big) = 0$

    in symbols $x_n \xrightarrow[w]{n\to\infty} x$

P.13. Lemma $(x_n)_n \subset X$ and $x \in X$

$$x_n \xrightarrow[w]{n\to\infty} x \text{ and } \|x_n\| \xrightarrow{n\to\infty} \|x\| \implies x_n \xrightarrow{n\to\infty} x$$

P.14. Example let $p \in [1, \infty]$

ℓ^p-spaces

$$\ell^p := \left\{ x = (x_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|x\|_p < \infty \right\}$$

$$\text{norm } \|x\|_p := \begin{cases} \left(\sum_{k\in\mathbb{N}} |x_k|^p\right)^{1/p} & \text{for } p \in [1,\infty[\\ \sup_{k\in\mathbb{N}} |x_k| & \text{for } p = \infty \end{cases}$$

($\|\cdot\|_p$ is a norm follows from Lemma P.17(b) as special case)

P.15. Definition (Lebesgue spaces) Let $p \in [1,\infty]$, $M \neq \emptyset$ a set and μ a measure on σ-alg. $\mathcal{A} \subseteq \mathcal{P}(M)$.

- $$\mathcal{L}^p(M) := \mathcal{L}^p(M,\mu) := \{ f : M \to \mathbb{C} \text{ measurable s.t. } \|f\|_p < \infty \}$$

$$\|f\|_p := \begin{cases} \left(\int_M d\mu \, |f|^p\right)^{1/p} & \text{for } p \in [1,\infty[\\ \operatorname{ess\,sup}_{x \in M} |f(x)| := \inf_{\substack{N \in \mathcal{A} \\ \mu(N)=0}} \sup_{x \in M \setminus N} |f(x)| & \text{for } p = \infty \end{cases}$$

- equivalence rel$^{\text{n}}$ on $\mathcal{L}^p$: $f \sim g \;:\Leftrightarrow\; f = g$ μ-a.e.

  equivalence class of f: $[f] := \{ g : g \sim f \}$

- $$L^p(M) := L^p(M,\mu) := \{ \underbrace{[f]}_{\equiv f \;\longleftarrow\; \text{(bad) convention!}} : f \in \mathcal{L}^p(M,\mu) \}$$

P.16. Examples

- $M = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$, $\mu = \mu_\# := \sum_{k \in \mathbb{N}} \delta_k$

$$\Rightarrow \quad f : M \to \mathbb{C} \longleftrightarrow (f(k))_{k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$$

$$\int_M d\mu \, f = \sum_{k \in \mathbb{N}} f(k), \qquad [f] = f$$

$$\Rightarrow \quad L^p(M,\mu_\#) = \ell^p$$

- $M = \Omega$ open subset of $\mathbb{R}^d$, $\mathcal{A} = \mathcal{B}(\mathbb{R}^d)$, $\mu = \lambda^d$

$\rightsquigarrow$ (standard) Lebesgue spaces

P.17. Lemma Let $f, g : M \to \mathbb{C}$ measurable

(a) (Genoal.) Hölder inequality $\forall r, p, q \in [1, \infty] : \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$

$$\|fg\|_r \leq \|f\|_p \|g\|_q$$

(special case $r = 1$, $p = q = 2$: Cauchy-Schwarz inequ.)

(b) Minkowski inequality $\forall p \in [1, \infty]$

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

P.18. Theorem (Riesz - Fischer)

$L^p(M, \mu)$ is a Banach space $\forall p \in [1, \infty[$

P.19. Theorem (Riesz representⁿ of dual space)

Let $p \in [1, \infty[$ and $1/p + 1/q = 1$. If $p = 1$ $(q = \infty)$ assume μ is σ-finite. Then

$$J : \begin{array}{l} L^q(M) \to (L^p(M))^* \\ f \mapsto l_f \end{array} \quad \text{with } l_f : \begin{array}{l} L^p(M) \to \mathbb{C} \\ g \mapsto \int_M d\mu\, fg \end{array}$$

is an isometric isomorphism, i.e. linear bijection with $\|f\|_q = \|l_f\|_{(L^p(M))^*}$

P.20. Remark If $p = \infty$ above, then J not surjective, i.e. $(L^\infty(M))^*$ is larger than $L^1(M)$

P.21. Example

Consider $(e^{(n)})_{n\in\mathbb{N}} \subset \ell^p$, $p\in[1,\infty]$, $e^{(n)}_k := \delta_{k,n}$ $\forall k,n\in\mathbb{N}$

$$\Rightarrow \|e^{(n)}-e^{(m)}\|_p = \begin{cases} 2^{1/p} & p\in[1,\infty[\\ 1 & p=\infty \end{cases} \quad \forall n\neq m\in\mathbb{N}$$

$\Rightarrow (e^{(n)})_n$ no Cauchy sequ. $\Rightarrow$ not (strongly) convergent

BUT: $\forall p\in]1,\infty[$ $e^{(n)} \xrightarrow[w]{n\to\infty} 0$ because

$\forall \ell\in(\mathcal{L}^p(M))^* \ \exists y\in\mathcal{L}^q(M)$, $1/q+1/p=1$: $\ell=\ell_y$ and

$$\ell_y(e^{(n)}) = \sum_{k\in\mathbb{N}} y_k e^{(n)}_k = y_n \xrightarrow{n\to\infty} 0$$

$q\in]1,\infty[$
$\left(\sum_{n\in\mathbb{N}}|y_n|^q\right)^{1/q}<\infty$

[obviously wrong for $p=1$: $y=(1,1,\dots)\in\ell^\infty$]

$\boxed{\text{P.22. Theorem}}$ Let $p\in[1,\infty[$ and $\emptyset\neq\Omega\subseteq\mathbb{R}^d$ open.

Then ℓ^p and $\mathcal{L}^p(\Omega,\lambda^d)\equiv\mathcal{L}^p(\Omega)$ are separable.

ℓ^∞, $\mathcal{L}^\infty(\Omega)$ are not separable.

Proof of separab. relies on

$$\left\{\sum_{n=1}^N q_n e^{(n)} : q_n\in\mathbb{Q}+i\mathbb{Q},\ N\in\mathbb{N}\right\} \text{ dense w.r.t. } \|\cdot\|_p \text{ in } \ell_p$$

for $p\in[1,\infty[$ and for $\mathcal{L}^p(\Omega)$ on

$\boxed{\text{P.23. Theorem}}$ Let $p\in[1,\infty[$ and $\emptyset\neq\Omega\subseteq\mathbb{R}^d$ open.

Then $C_c^\infty(\Omega) := \{f:\Omega\to\mathbb{C}$ arbitrarily often different. and supp f compact in $\Omega\}$ $\subset C^\infty(\Omega)$

is dense in $\mathcal{L}^p(\Omega)$ (w.r.t. $\|\cdot\|_p$)

P.24. Remark

$$\overline{C_c^\infty(\Omega)}^{\|\cdot\|_\infty} = C_0^\infty(\Omega) := \{ f \in C^\infty(\Omega) \text{ and } \forall \varepsilon > 0 \ \exists K_\varepsilon \subseteq \Omega \text{ cpt. s.t. } |f(x)| \leq \varepsilon \ \forall x \in \Omega \setminus K_\varepsilon \}$$

vanish towards the boundary

P.25. Theorem (p.t.o.)

P.26. Definition

$\mathcal{H}$ Hilbert space $:\Leftrightarrow$ $\mathcal{H}$ Banach space and $\|x\| = \sqrt{\langle x,x\rangle}$ $\forall x \in \mathcal{H}$

where $\langle \cdot,\cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ non-degenerate sesquilinear form (scalar product)

convention: $\langle \cdot,\cdot \rangle$ anti-linear in left argument

P.27. Lemma (Riesz representation)

$$J : \mathcal{H} \to \mathcal{H}^*, \quad \psi \mapsto \langle \psi, \cdot \rangle$$ is antilinear isometric isomorphism

$\|\psi\| = \| \langle \psi, \cdot \rangle \|_*$

P.28. Corollary $(\varphi_n)_n \subset \mathcal{H}$, $\varphi \in \mathcal{H}$. Then

$$\varphi_n \xrightarrow[w]{n\to\infty} \varphi \iff \forall \psi \in \mathcal{H} : \langle \psi, \varphi_n \rangle \xrightarrow{n\to\infty} \langle \psi, \varphi \rangle$$

P.29. Lemma $\mathcal{H}$ Hilbert space

(a) $\mathcal{H}$ separable $\iff$ $\exists$ countable ONB $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$

(b) $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ pairwise orthogonal. Then

$$\sum_{n\in\mathbb{N}} \|\varphi_n\|^2 < \infty \iff \sum_{n\in\mathbb{N}} \varphi_n \text{ converges in } \mathcal{H}$$

P.25. Theorem (Fundamental lemma of the calculus of variations)

Let $\psi \in L^1_{loc}(\mathbb{R}^d) := \{ \psi : \mathbb{R}^d \to \mathbb{C} \text{ measurable s.t. } \psi 1_K \in L^1(\mathbb{R}^d)\ \forall K \subset \mathbb{R}^d \text{ compact} \}$

If
$$\int_{\mathbb{R}^d} dx\ \psi(x)\, f(x) = 0 \qquad \forall f \in C_c^\infty(\mathbb{R}^d)$$

Then $\psi(x) = 0$ for a.e. $x \in \mathbb{R}^d$. If even $\psi \in C(\mathbb{R}^d)$ ($\Rightarrow L^1_{loc}$), then $\psi(x) = 0\ \forall x \in \mathbb{R}^d$.

(cf. Thm. P.4(c))

P.3. Bounded linear operators

[Literature: see P.2]

X, Y normed spaces, $\mathcal{H}$ Hilbert space

P.30. Definition Let $X_0 \subseteq X$ be a dense subspace (domain) and $A : X_0 \to Y$ linear.

$$A \text{ bounded} :\Leftrightarrow \|A\| := \sup_{0 \neq \psi \in X_0} \frac{\|A\psi\|_{(Y)}}{\|\psi\|_{(X)}} < \infty$$

Note $\|A\psi\| \leq \|A\| \, \|\psi\| \; \forall \psi \in X_0$

A unbounded $:\Leftrightarrow A$ not bdd.

P.31. Theorem (Bounded extension)

(a) Let $A : X_0 \to Y$ be a bdd. (linear) op. Then $\exists_1 \hat{A} : X \to Y$ linear s.t. $\hat{A}|_{X_0} = A$ and $\|\hat{A}\| = \|A\|$.

[We identify them and write again A for $\hat{A}$]

(b) Let $BL(X,Y) := \{A : X \to Y \text{ linear and bounded}\}$. Then $BL(X,Y)$ is a normed vector space. If Y is complete, so is $BL(X,Y)$.

P.32. Theorem Let $A : X_0 \to Y$ linear. The following are equivalent

(i) A continuous

(ii) $\exists \psi \in X : A$ continuous in ψ

(iii) A bdd.

P.33. Definition Let $(A_n)_n \subset BL(X,Y)$, $A \in BL(X,Y)$.

- $(A_n)_n$ converges to A (in norm, in $BL(X,Y)$, uniformly) $:\Leftrightarrow \lim_{n\to\infty} \|A_n - A\| = 0$

  in symbols: $A = \lim_{n\to\infty} A_n$

- $(A_n)_n$ converges strongly to A $:\Leftrightarrow A_n\psi \xrightarrow{n\to\infty} A\psi$ (in Y) $\forall \psi \in X$

  in symbols: $A_n \underset{s}{\xrightarrow{n\to\infty}} A$

- $(A_n)_n$ converges weakly to A $:\Leftrightarrow A_n\psi \underset{w}{\xrightarrow{n\to\infty}} A\psi$ (in Y) $\forall \psi \in X$

  special case $X = Y = \mathcal{H}$: $\forall \varphi, \psi \in \mathcal{H}$

  $\lim_{n\to\infty} \langle \varphi, A_n\psi\rangle = \langle \varphi, A\psi\rangle$

P.34. Lemma norm conv. $\Rightarrow$ strong conv. $\Rightarrow$ weak conv.

P.35. Definition X Banach space, $A \in BL(X)$

- resolvent set $\rho(A) := \{ z \in \mathbb{C} : A - \underbrace{z\mathbb{1}}_{=:z} \text{ bijective} \}$
- spectrum $\operatorname{spec}(A) := \sigma(A) := \mathbb{R} \setminus \rho(A)$
- point spectrum

  $\operatorname{spec}_p(A) := \sigma_p(A) := \{ z \in \mathbb{C} ; A - z \text{ not injective} \} = \{ \text{eigenvalues of } A \}$

- continuous spectrum

  $\operatorname{spec}_c(A) := \sigma_c(A) := \{ z \in \mathbb{C} : A - z \text{ injective but not surjective and } \operatorname{ran}(A-z) \text{ dense in } X \}$

- residual spectrum

$$\mathrm{spec}_r(A) := \sigma_r(A) = \{ z \in \mathbb{C} : A - z \text{ not injective and } \mathrm{ran}\,(A-z) \text{ not dense} \}$$

- resolvent (Green function)

$R_z := (A-z)^{-1}$ for $z \in \mathbb{C}$ for which it exists as densely def. operator, i.e. for $z \in \rho(A) \cup \sigma_c(A)$

P.36 Remark

(a) $\sigma(A) = \sigma_p(A) \,\dot{\cup}\, \sigma_c(A) \,\dot{\cup}\, \sigma_r(A)$

(b) $\sigma_r(A) = \emptyset$ for A of interest

(c) If $\dim X < \infty \Rightarrow \sigma_c(A) = \sigma_r(A) = \emptyset$

(d) Bdd. Inv. Thm: X, Y Banach, $B \in BL(X,Y)$ bijective. Then $B^{-1} \in BL(Y,X)$

$\Rightarrow R_z \in BL(X)$ for $z \in \rho(A)$

$\boxed{\text{P.37. Theorem}}$ X Banach, $A \in BL(X)$. Then

(a) $\sigma(A) \neq \emptyset$

(b) $\rho(A)$ open ($\Rightarrow \sigma(A)$ closed)

(c) $\rho(A) \ni z \longmapsto R_z$ is strongly analytic

For unbdd. operators R_z need not be bdd. for $z \in \rho(A)$ in general. But will be true for closed operators on a Hilbert space ...