Sharp Semiclassical Spectral Estimates with Remainder Terms

Spectral Days 2012, München

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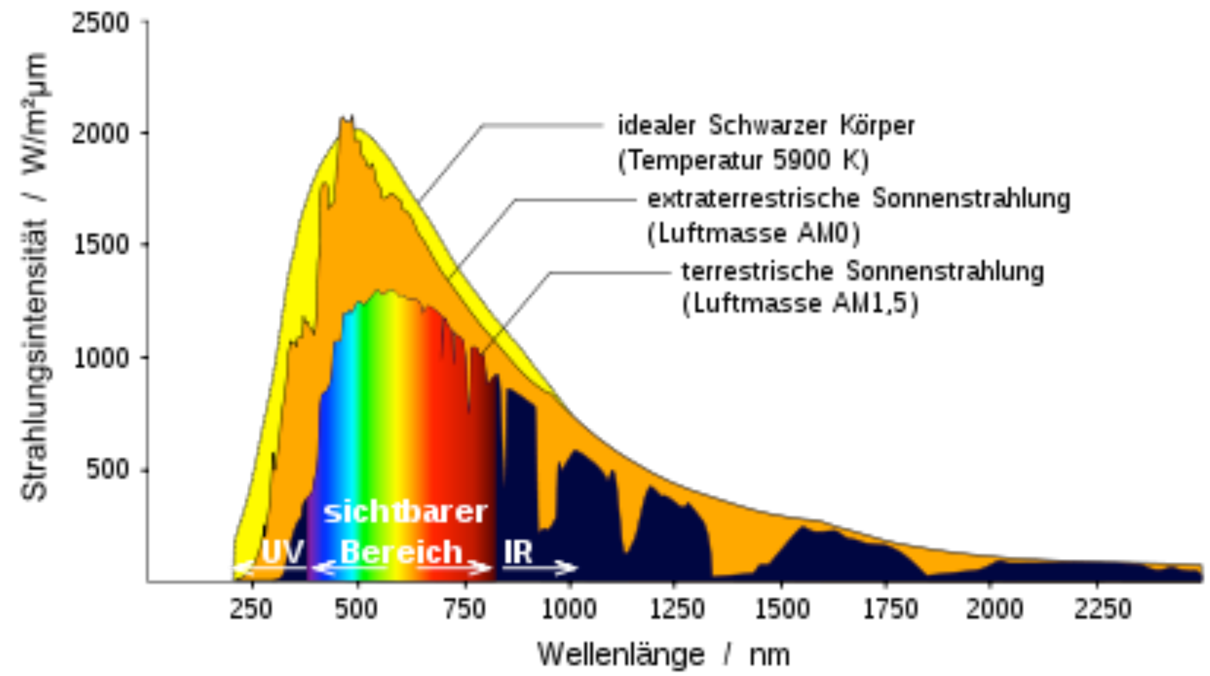
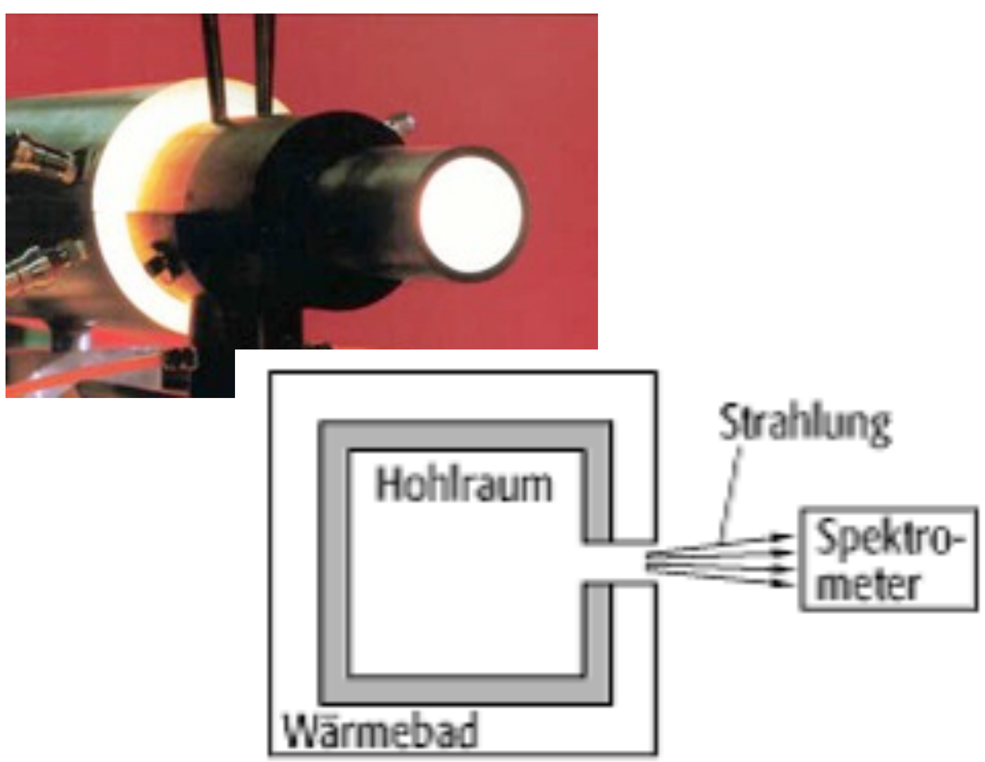
Carl Friedrich von Siemens Stiftung

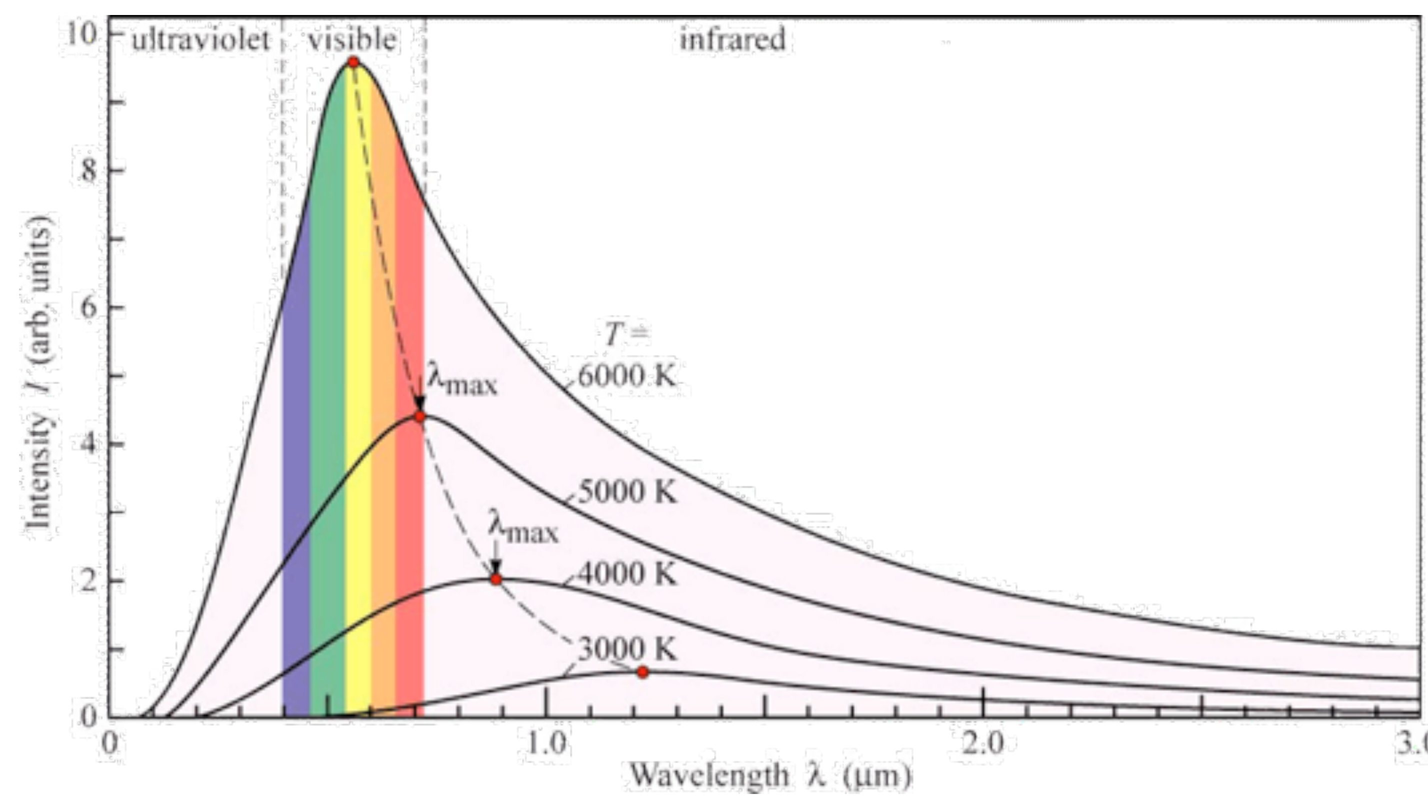
$\int \mathbf{A} d\mu$

13.4.2012

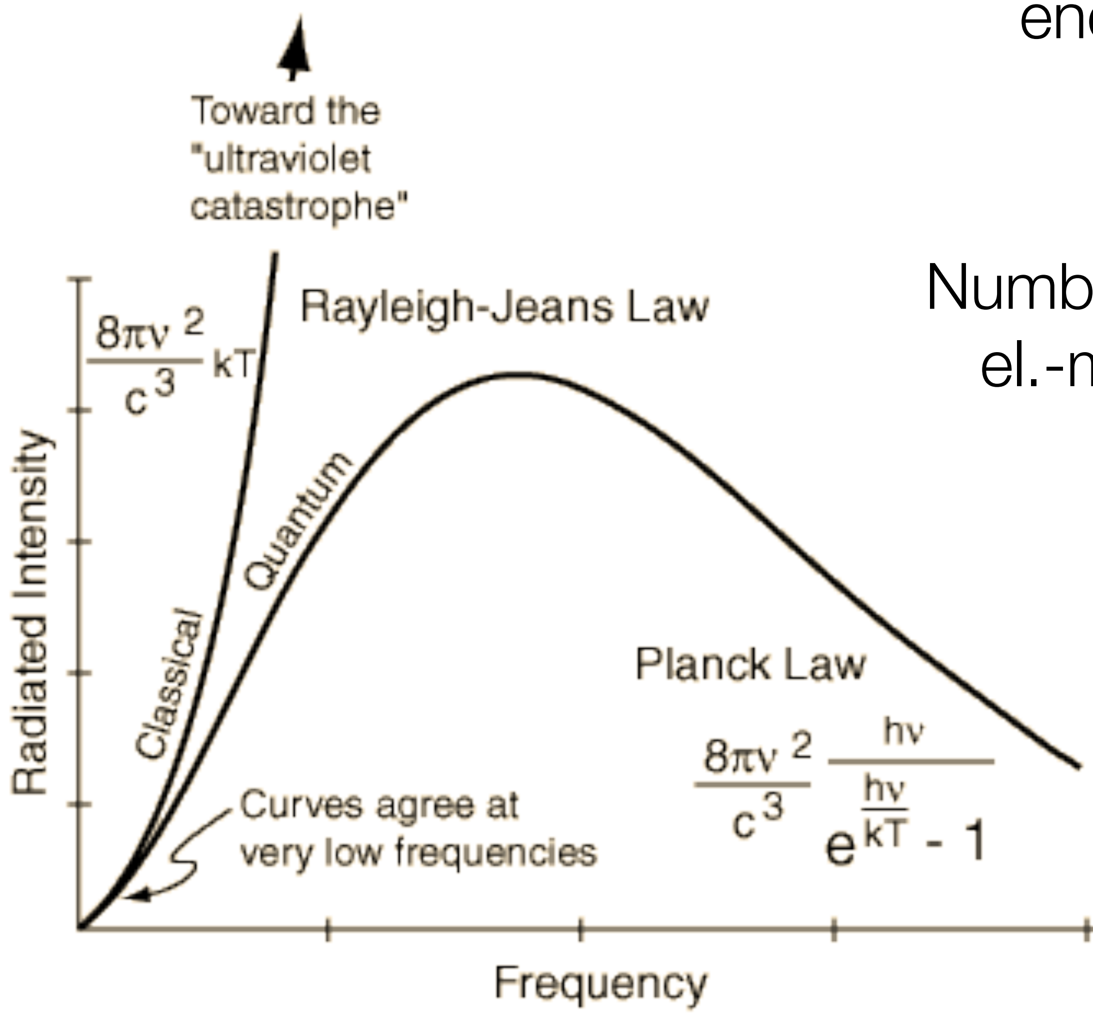
The colour of the sun
The sun - a black body?

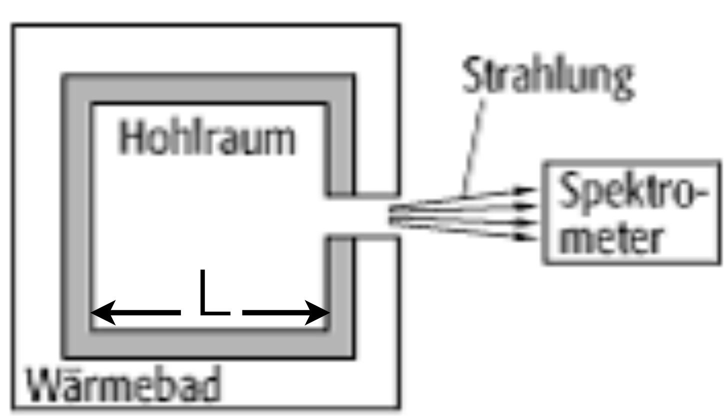






Rayleigh-Jeans` and Planck`s radiation law





energy of el.-m. radiation in a cube of length L in the frequency interval $d\nu$ is given by

Number of modes dN of the el.-magn. resonator in the frequency range $[\nu, \nu + d\nu]$ $\times$ Average energy $E(\nu)$ at the frequency ν

$$dN \times E(\nu)$$

Rayleigh-Jeans $E(\nu) = kT$

Planck $E(\nu) = \dfrac{h\nu}{e^{\frac{h\nu}{kT}} - 1}$

$$dN = \frac{8\pi\nu^2 d\nu}{c^3} L^3$$

Planck`s radiation law

sparks theory of spectral asymptotics

sparks Quantum Mechanics

$$\frac{8\pi\nu^2 d\nu}{c^3} L^3 \times \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}$$

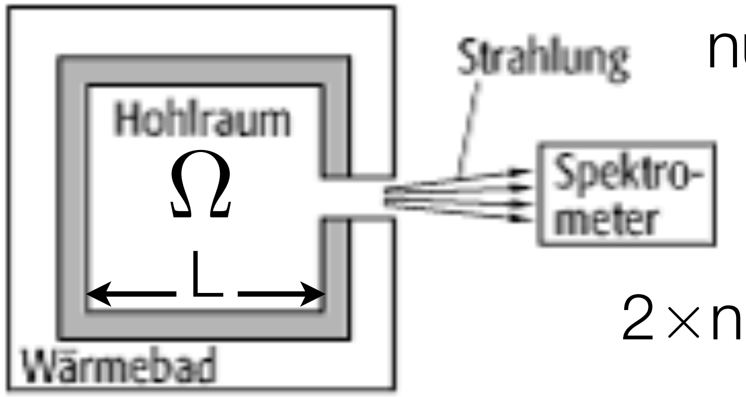
universal radiation law

classification of stars, 3K background radiation

$$dN = \frac{8\pi\nu^2 d\nu}{c^3} L^3$$

corresponds to

$$N = \frac{8\pi\nu^3}{3c^2} L^3 = 2 \times \frac{4\pi}{3} \frac{1}{(2\pi)^3} \Lambda^{\frac{3}{2}} \mathrm{vol}(\Omega)$$

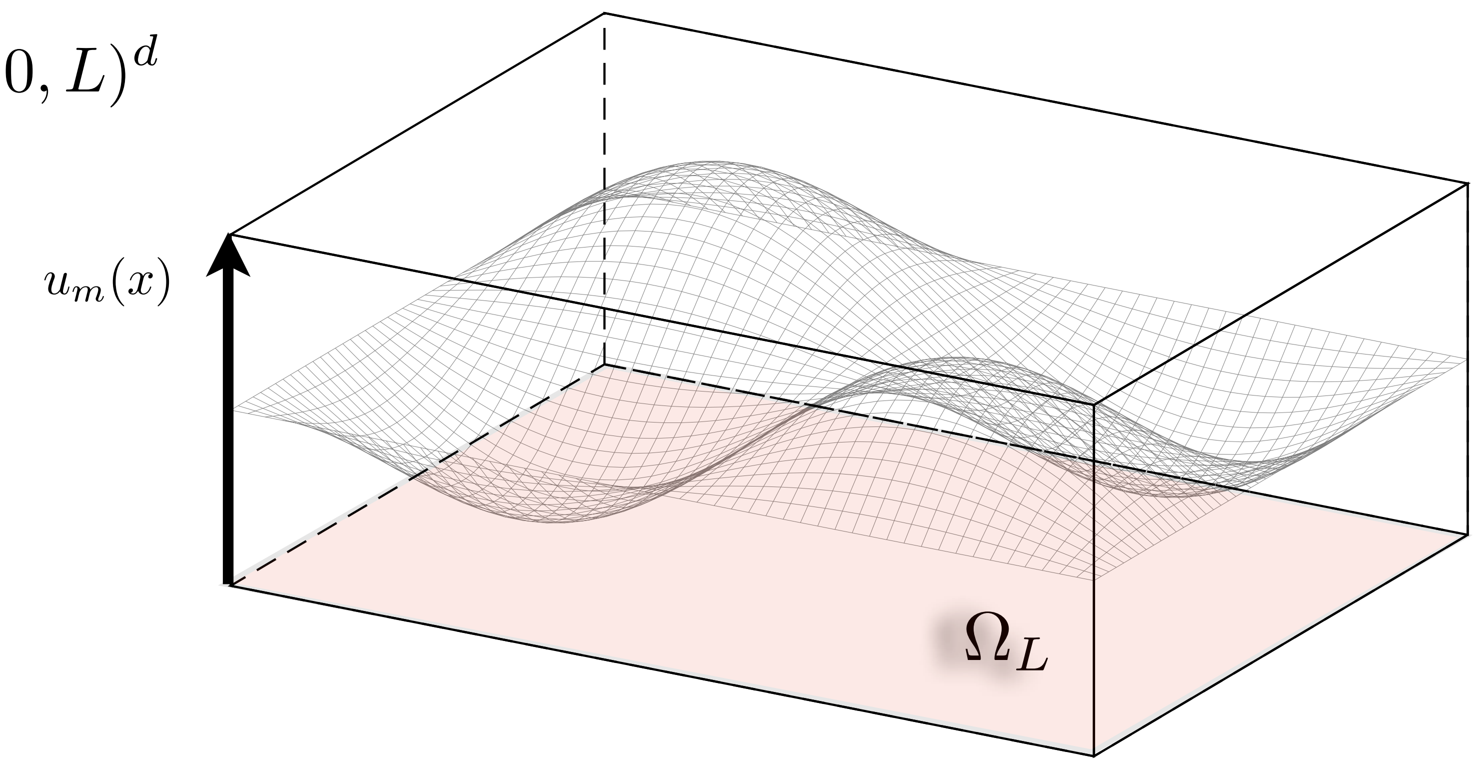


number of modes with frequencies not exceeding ν in the cube Ω

$2\times$number of the eigenvalues of the Dirichlet Laplacian in the cube Ω not exceeding $\Lambda = 4\pi^2\nu^2c^{-2}$.

Modes in a cubic resonator

$$\Omega = \Omega_L = (0,L)^d$$



$$-\Delta u_m(x) = \lambda_m^D(\Omega_L) u_m(x)$$

$$u_m(x) = \prod_{k=1}^{d} \sin\left(\frac{\pi}{L} m_k x_k\right), \qquad \lambda_m^D(\Omega_L) = \frac{\pi^2}{L^2}\sum_{k=1}^{d} m_k^2, \qquad m_k = 1, 2, \dots,$$

Number of modes in a cube

The Dirichlet or Neumann eigenfunction in the cube $\Omega_L = \{0 < x_k < L \mid k = 1, \dots, d\}$

are given by

$$u_m(x) = \prod_{k=1}^{d} \sin\left(\frac{\pi}{L} m_k x_k\right), \qquad \lambda_m^D(\Omega_L) = \tfrac{\pi^2}{L^2} \textstyle\sum_{k=1}^d m_k^2, \qquad m_k = 1, 2, \dots,$$

$$u_m(x) = \prod_{k=1}^{d} \cos\left(\frac{\pi}{L} m_k x_k\right), \qquad \lambda_m^N(\Omega_L) = \tfrac{\pi^2}{L^2} \textstyle\sum_{k=1}^d m_k^2, \qquad m_k = 0, 1, \dots,$$

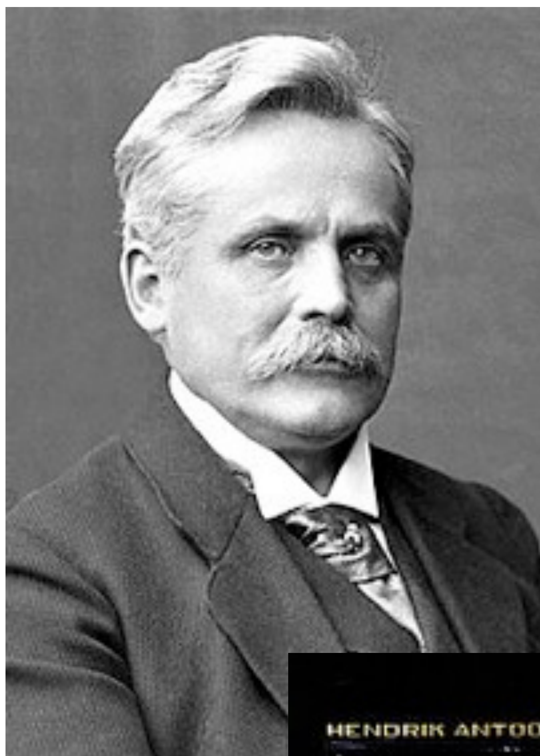
Hence

$$\begin{aligned} n_{D,N}^{\Omega_L}(\Lambda) &= \# \left\{ m \in \mathbb{Z}^d \;\middle|\; \|m\|^2 \le \frac{L^2}{\pi^2}\Lambda, \quad \begin{matrix} m_k \ge 1 & \text{DRB} \\ m_k \ge 0 & \text{NRB} \end{matrix} \right\} \\ &= \frac{\tau_d}{2^d} \left(\frac{L}{\pi} \Lambda^{\frac{1}{2}} \right)^d (1 + o(1)) = \frac{\tau_d}{(2\pi)^d} \cdot \Lambda^{\frac{d}{2}} \cdot \mathrm{vol}(\Omega_L) \cdot (1 + o(1)) \\ &= \int\int_{x \in \Omega, \|\xi\|^2 \le \Lambda} \frac{dx \cdot d\xi}{(2\pi)^d} (1 + o(1)) \quad \text{as} \quad \Lambda \to +\infty \end{aligned}$$

τ_d Volume of the unit ball in the dimension d

A bit of history

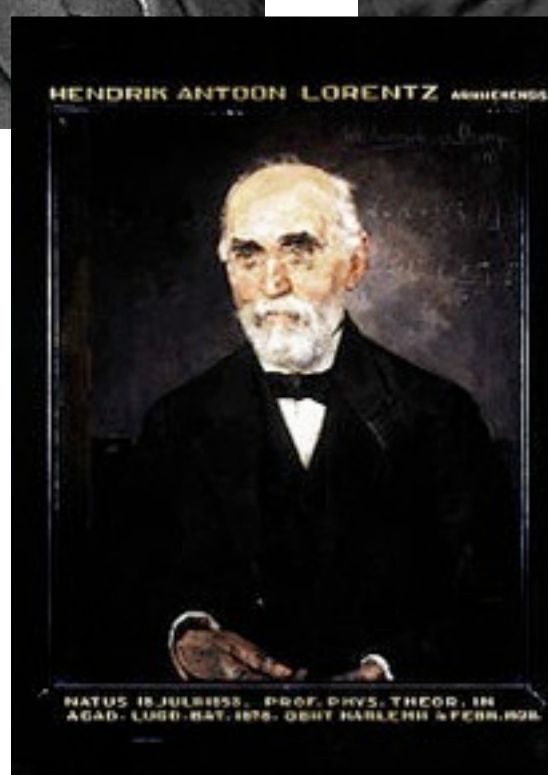
W. Wien, NP 1911



Baron Rayleigh, NP 1904

M. Planck, NP 1918

Sir J. Jeans

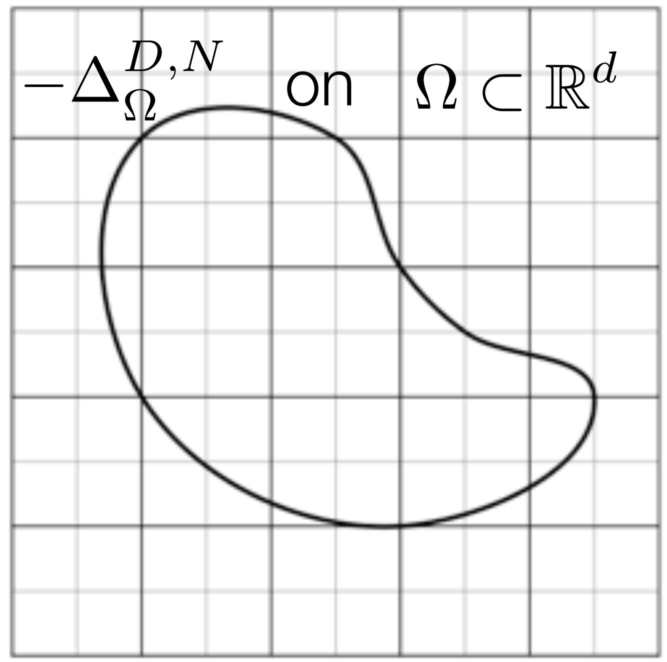


H. A. Lorentz, NP 1902

D. Hilbert

H. Weyl

100th anniversary of Weyl`s law



H. Weyl: Über die Abhängigkeit der Eigenschwingungen einer Membran von deren Begrenzung, Journal für die reine und angewandte Mathematik 141 (1912) 1–11.

H. Weyl: Über das Spektrum der Hohlraumstrahlung." Journal für die reine und angewandte Mathematik 141 (1912) 163–181.

$$n_{D,N}^{\Omega}(\Lambda) = \frac{\tau_d}{(2\pi)^d} \cdot \Lambda^{\frac{d}{2}} \cdot \mathrm{vol}(\Omega) + o(\Lambda^{\frac{d}{2}}) \quad \text{as} \quad \Lambda \to +\infty$$

1911-1912

$$n_{D,N}^{\Omega}(\Lambda) = \frac{\tau_d}{(2\pi)^d} \cdot \Lambda^{\frac{d}{2}} \cdot \mathrm{vol}(\Omega) \mp \frac{1}{4} \frac{\tau_{d-1}}{(2\pi)^{d-1}} \Lambda^{\frac{d-1}{2}} |\partial\Omega| + o(\Lambda^{\frac{d-1}{2}}) \quad \text{as} \quad \Lambda \to +\infty$$

under additional geometrical conditions, V. Ivrii in 1980

50th anniversary of Pólya`s conjecture

Weyl: $$n^{\Omega}_{D,N}(\Lambda) = \frac{\tau_d}{(2\pi)^d} \cdot \Lambda^{\frac{d}{2}} \cdot \mathrm{vol}(\Omega) + o(\Lambda^{\frac{d}{2}}) \quad \text{as} \quad \Lambda \to +\infty$$

G. Pólya: “On the eigenvalues of a vibrating membrane”, Proc. London Math. Soc. 11 (1961) 419– 433.

He proves in a sublime way an amazing bound for **tiling** domains:

$$n^{\Omega}_{D}(\Lambda) \leq \frac{\tau_d}{(2\pi)^d} \cdot \Lambda^{\frac{d}{2}} \cdot \mathrm{vol}(\Omega) \quad \text{for all} \quad \Lambda > 0$$

Pólya conjectures this bound to be true for **all** domains.

This conjecture is still unresolved!

It is supported by the second term in the Weyl asymptotics

$$n^{\Omega}_{D}(\Lambda) = \frac{\tau_d}{(2\pi)^d} \cdot \Lambda^{\frac{d}{2}} \cdot \mathrm{vol}(\Omega) - \frac{1}{4}\frac{\tau_{d-1}}{(2\pi)^{d-1}}\Lambda^{\frac{d-1}{2}}|\partial\Omega| + o(\Lambda^{\frac{d-1}{2}}) \quad \text{as} \quad \Lambda \to +\infty$$

Some (almost) new results
Pólya`s conjecture fails for magnetic fields!

Replace $-\Delta_\Omega^D$ by $(i\nabla + A(x))^2_{D,\Omega}$, where $A(x)$ is a magnetic vector potential.

Since

$$\int\int_{x\in\Omega,\|\xi\|^2\leq\Lambda} \frac{dx\cdot d\xi}{(2\pi)^d} = \int\int_{x\in\Omega,\|\xi+A(x)\|^2\leq\Lambda} \frac{dx\cdot d\xi}{(2\pi)^d}$$

the phase space volume does not change and, moreover, the Weyl asymptotics

$$n_D^\Omega(\Lambda;A) = \frac{\tau_d}{(2\pi)^d}\cdot\Lambda^{\frac{d}{2}}\cdot\mathrm{vol}(\Omega)\cdot(1+o(1)) \quad \text{as} \quad \Lambda\to+\infty$$

remains true (under mild regularity conditions on A).

At the same time for constant magnetic fields $A(x) = 2^{-1}B(x_2,-x_1)$

we prove a modified 2d-Pólya bound with a sharp constant

$$n_D^\Omega(\Lambda;A) \leq 2\frac{\tau_2}{(2\pi)^2}\cdot\Lambda\cdot\mathrm{vol}(\Omega) \quad \text{for all} \quad \Lambda>0$$

R. Frank
M. Loss
T.W.
2008

This constant can be approached on squares with properly scaled Λ and B .
Hence, it cannot be improved even for tiling domains.

Riesz means of eigenvalues

Alongside with the individual eigenvalues $\lambda_m^D(\Omega)$ it is useful to consider the Riesz means

$$S_{\sigma,d}(\Lambda,\Omega) = \sum_m (\Lambda - \lambda_m^D(\Omega))_+^\sigma, \quad \sigma \ge 0\,.$$

and compare it with the phase space average

$$S^{\mathrm{cl}}_{\sigma,d}(\Lambda,\Omega) = \int\int_{x\in\Omega, \|\xi\|^2\le\Lambda} (\Lambda - |\xi|^2)_+^\sigma \frac{dx\cdot d\xi}{(2\pi)^d} = L^{\mathrm{cl}}_{\sigma,d}\Lambda^{\sigma+\frac{d}{2}}\mathrm{vol}(\Omega)$$

where the classical Lieb-Thirring constant $L^{\mathrm{cl}}_{\sigma,d}$ equals to $\dfrac{\Gamma(\sigma+1)}{2^d\pi^{d/2}\Gamma\left(\sigma+\frac{d}{2}+1\right)}$.

Integrating the Weyl asymptotics for the counting function one has

$$\sum_m (\Lambda - \lambda_m^D(\Omega))_+^\sigma = S_{\sigma,d}(\Lambda,\Omega) = S^{\mathrm{cl}}_{\sigma,d}(\Lambda,\Omega)(1+o(1)) = L^{\mathrm{cl}}_{\sigma,d}\Lambda^{\sigma+\frac{d}{2}}\mathrm{vol}(\Omega)(1+o(1))$$

for $\Lambda \to +\infty$.

The case $\sigma = 0$ corresponds to the counting function $n_D^\Omega(\Lambda) = S_{0,d}(\Lambda,\Omega)$.

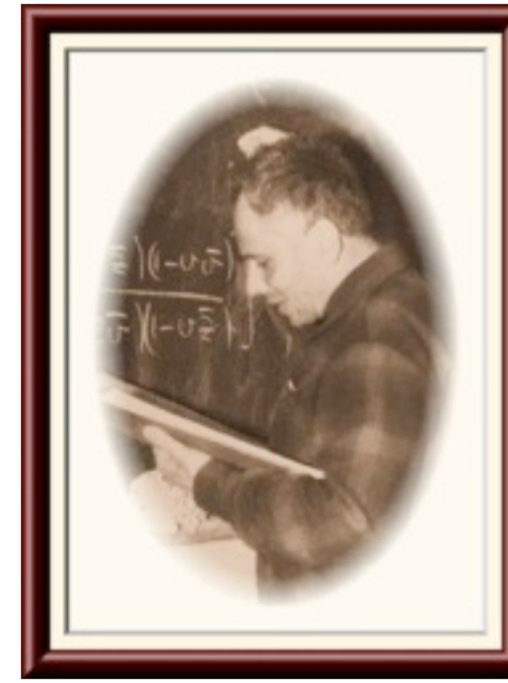
40th anniversary of Berezin`s bound

Pólyas (non-magnetic) bound implies

$$S_{\sigma,d}(\Lambda,\Omega) \leq S^{\mathrm{cl}}_{\sigma,d}(\Lambda,\Omega)$$

for all **tiling** domains Ω and all $\sigma, \Lambda > 0 \cdot$

Berezin proved 1972 this bound for **all** domains Ω and $\sigma \geq 1, \Lambda > 0$



One can now ask, whether a bound

$$S_{\sigma,d}(\Lambda,\Omega) \leq c(\sigma,d) S^{\mathrm{cl}}_{\sigma,d}(\Lambda,\Omega)$$

holds true for **all** domains?

His result implies also this bound with (probably) non-sharp constants $c(\sigma,d)$ for all $0 \leq \sigma < 1, \Lambda > 0.$

40th anniversary of Berezin`s bound

$S_{\sigma,d}(\Lambda,\Omega,A) \le c(\sigma,d) S^{\mathrm{cl}}_{\sigma,d}(\Lambda,\Omega)$

If one allows for constant magnetic fields $A(x) = 2^{-1}B(x_2,-x_1)$, then

$$c(\sigma,2) = 2\left(\frac{\sigma}{1+\sigma}\right)^{\sigma} > 1 \quad \text{for} \quad 0 \le \sigma < 1$$

Frank, Loss, W.

is the best possible constant, which is independent on the strength of A.

Moreover, if the domain is tiling, then

$$S_{\sigma,2}(\Omega,\Lambda,\mathcal{A}) \le \mathfrak{B}_\sigma(B,\Lambda)\mathrm{vol}(\Omega)\,, \quad \mathfrak{B}_\sigma(B,\Lambda) = \frac{B}{2\pi}\sum_{k\ge 0}(\Lambda - B(2k+1))_+^\sigma\,.$$

Moreover, for constant magnetic fields one has $c(\sigma,d) = 1$ for all $\sigma \ge 1.$ Erdös, Loss, Wugalter

Finally, for arbitrary magnetic fields one has $c(\sigma,d) = 1$ for all $\sigma \ge \frac{3}{2}.$ Laptev, W., Helffer

$$S_{1,d}(\Lambda,\Omega) \leq S^{\mathrm{cl}}_{1,d}(\Lambda,\Omega)$$

Berezin`s proof

Put $\sigma = 1.$ Let $\{\psi_k\}$ be the ONB of eigenfunctions of $-\Delta^D_\Omega$ corresponding to $\{\lambda^D_k(\Omega)\}$.
Extend these Dirichlet eigenfunctions by zero to $\mathbb{R}^d$ and let $\{\hat\psi_k\}$ be the (ON) FT.

$$\sum_{k:\lambda^D_k<\Lambda} (\Lambda - \lambda^D_k) = \sum_{k:\lambda^D_k<\Lambda} \left(\Lambda - \int_{\mathbb{R}^d} |\nabla\psi_k|^2 dx\right) = \sum_{k:\lambda^D_k<\Lambda} \left(\Lambda - \int_{\mathbb{R}^d} \|\xi\|^2 |\hat\psi_k|^2 d\xi\right)$$

Since $\int_{\mathbb{R}^d} |\hat\psi_k|^2 d\xi = 1$ we have

$$\sum_{k:\lambda^D_k<\Lambda} (\Lambda - \lambda^D_k) = \sum_{k:\lambda^D_k<\Lambda} \int_{\mathbb{R}^d} (\Lambda - \|\xi\|^2)|\hat\psi_k|^2 d\xi = \int_{\|\xi\|^2<\Lambda} (\Lambda - \|\xi\|^2) \sum_k |\hat\psi_k|^2 d\xi$$

$$+ \sum_{k:\lambda^D_k<\Lambda} \int_{\|\xi\|^2\geq\Lambda} (\Lambda - \|\xi\|^2)|\hat\psi_k|^2 d\xi - \sum_{k:\lambda^D_k\geq\Lambda} \int_{\|\xi\|^2<\Lambda} (\Lambda - \|\xi\|^2)|\hat\psi_k|^2 d\xi$$

Since $\sum_k |\hat\psi_k|^2 = (2\pi)^{-d} \sum_k \left|\langle \psi_k, e^{i\xi\cdot}\rangle_{L^2(\Omega)}\right|^2 = (2\pi)^{-d} \left\|e^{i\xi\cdot}\right\|^2_{L^2(\Omega)} = (2\pi)^{-d}\mathrm{vol}(\Omega)$

$$S_{1,d}(\Lambda,\Omega) = \sum_{k:\lambda^D_k<\Lambda} (\Lambda - \lambda^D_k) = S^{cl}_{1,d}(\Lambda,\Omega) + r(\Lambda,\Omega), \quad r(\Lambda,\Omega) \leq 0.$$

Li-Yau, Melas and beyond

Let $\{\psi_k\}$ be the ONB of eigenfunctions of $-\Delta_\Omega^D$ corresponding to $\{\lambda_k^D(\Omega)\}$.
Extend these Dirichlet eigenfunctions by zero to $\mathbb{R}^d$ and let $\{\hat\psi_k\}$ be the (ON) FT.

$$F_N(\xi) = \sum_{j=1}^N |\hat\psi_j(\xi)|^2 \le (2\pi)^{-d} \|e^{i\xi x}\|^2_{L^2(\Omega)} = (2\pi)^{-d}\mathrm{vol}(\Omega)\,.$$

$$\int F_N(\xi)d\xi = N$$

We estimate $I(F_N) = \sum_{j=1}^N \lambda_j = \int_{\mathbb{R}^d} |\xi|^2 F_N(\xi)d\xi$

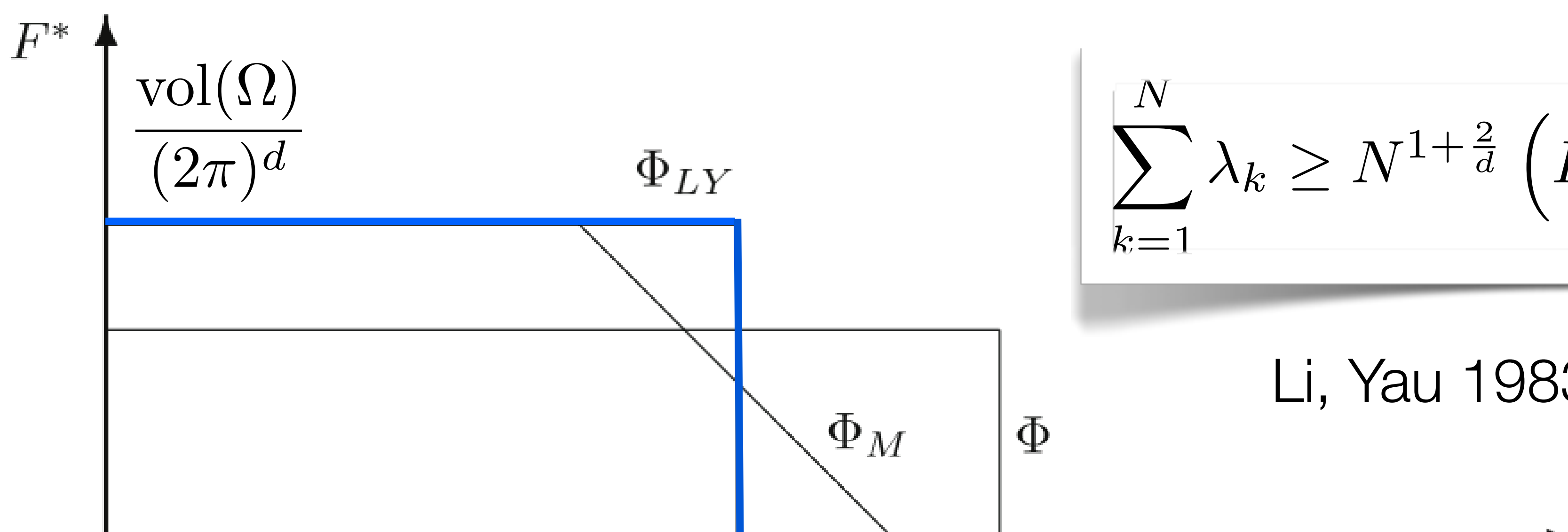


Fig. 1. Minimizers of the functional $\int_{\mathbb{R}^2} |\xi|^2 F^*(|\xi|)\, d\xi$

$$\sum_{k=1}^N \lambda_k \ge N^{1+\frac{2}{d}} \left(L^{\mathrm{cl}}_{0,d}\mathrm{vol}(\Omega)\right)^{-\frac{2}{d}} \frac{d}{2+d}$$

Li, Yau 1983

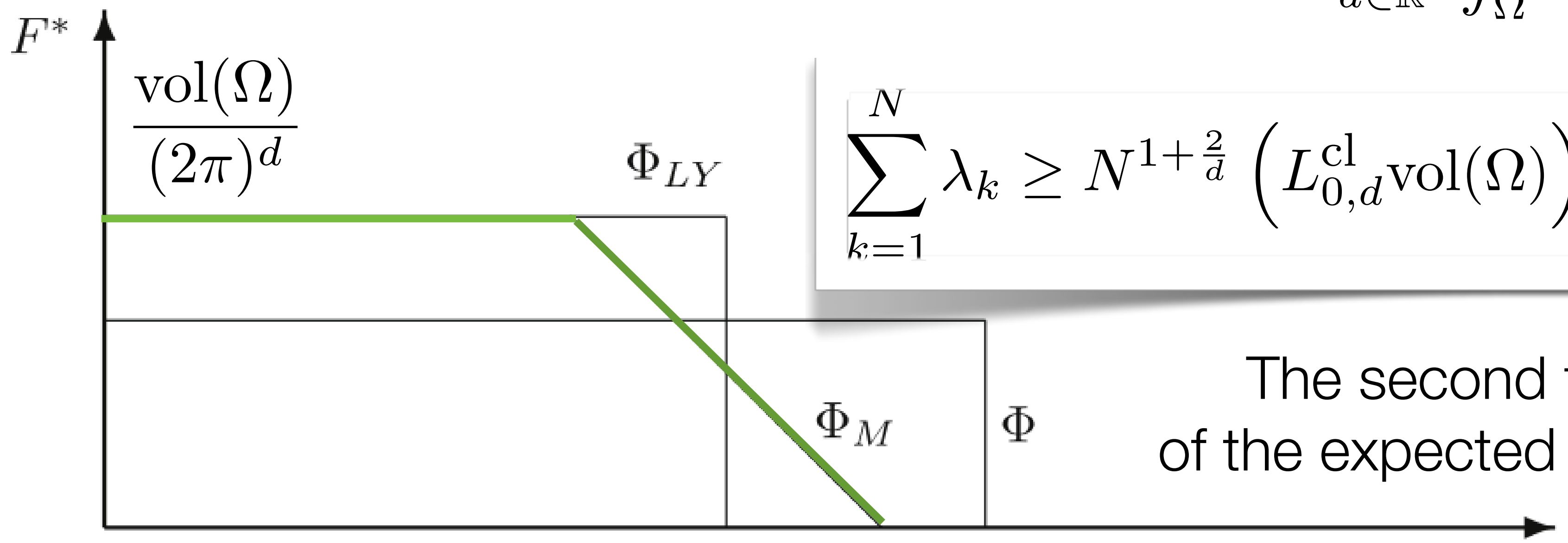
Li-Yau, Melas and beyond

We still estimate $I(F_N) = \sum_{j=1}^{N} \lambda_j = \int_{\mathbb{R}^d} |\xi|^2 F_N(\xi) d\xi$

$$F_N(\xi) = \sum_{j=1}^{N} |\hat{\psi}_j(\xi)|^2 \leq (2\pi)^{-d} \|e^{i\xi x}\|^2_{L^2(\Omega)} = (2\pi)^{-d} \mathrm{vol}(\Omega)\,. \qquad \int F_N(\xi) d\xi = N$$

Melas takes into account the following additional condition

$$|\nabla F_N| \leq 2(2\pi)^{-d} \sqrt{J(\Omega)\mathrm{vol}(\Omega)} \qquad J(\Omega) = \min_{a\in\mathbb{R}^d} \int_\Omega |x-a|^2 dx$$



$$\sum_{k=1}^{N} \lambda_k \geq N^{1+\frac{2}{d}} \left(L^{\mathrm{cl}}_{0,d} \mathrm{vol}(\Omega)\right)^{-\frac{2}{d}} \frac{d}{2+d} + N \cdot M_d \frac{\mathrm{vol}(\Omega)}{J(\Omega)}$$

The second term here is not of the expected second Weyl order $N^{1+\frac{1}{d}}$

$$\sum_{k=1}^{N} \lambda_k = \frac{d}{2+d} \left(L^{cl}_{0,d}\right)^{-\frac{2}{d}} (\mathrm{vol}(\Omega))^{-\frac{2}{d}} N^{1+\frac{2}{d}} + \frac{L^{cl}_{1,d-1}(L^{cl}_{1,d})^{-1-\frac{1}{d}}}{4(\frac{d-1}{2}+1)} \cdot \frac{|\partial\Omega|}{(\mathrm{vol}(\Omega))^{1+\frac{1}{d}}} N^{1+\frac{1}{d}} + o(N^{1+\frac{1}{d}})$$

Li-Yau, Melas and beyond

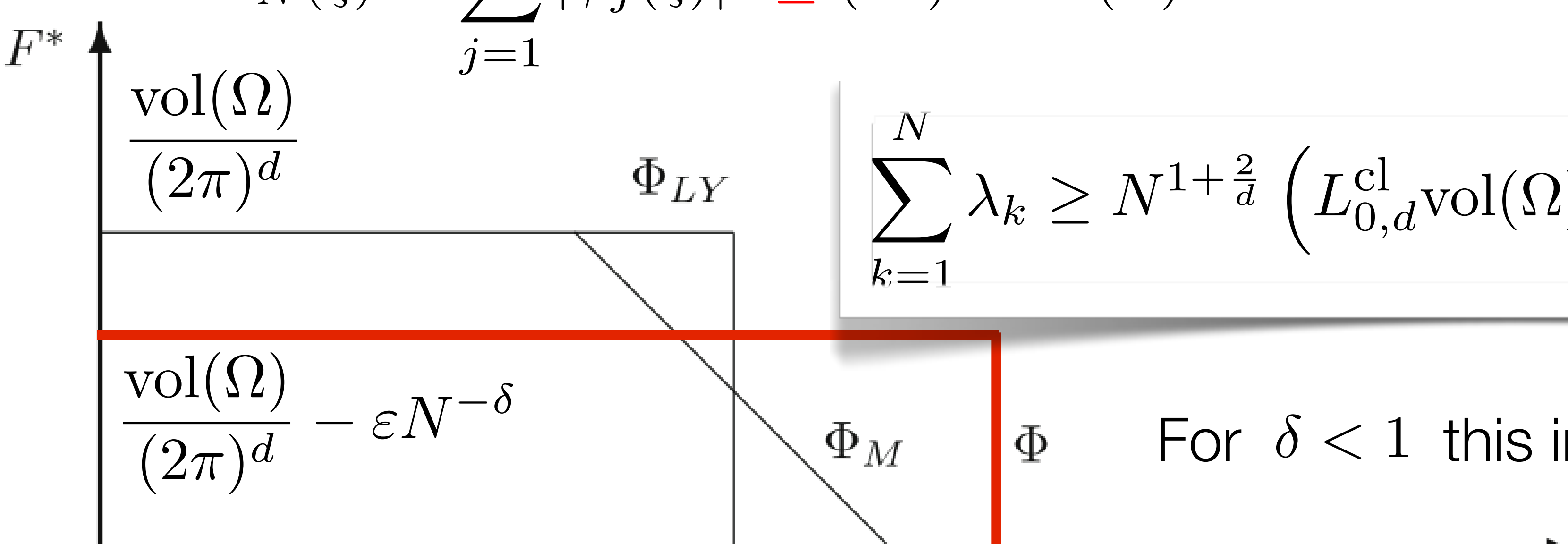
$$I(F_N) = \sum_{j=1}^{N} \lambda_j = \int_{\mathbb{R}^d} |\xi|^2 F_N(\xi)\, d\xi \qquad \int F_N(\xi)\, d\xi = N$$

$$F_N(\xi) = \sum_{j=1}^{N} |\hat{\psi}_j(\xi)|^2 \;\leq (2\pi)^{-d} \|e^{i\xi x}\|^2_{L^2(\Omega)} = (2\pi)^{-d} \mathrm{vol}(\Omega)$$

But that is picking the wrong fight. The true remainder term is hidden inBessel`s inequality

Assume we could quantify the remainder in Bessel`s inequality as follows

$$F_N(\xi) = \sum_{j=1}^{N} |\hat{\psi}_j(\xi)|^2 \leq (2\pi)^{-d} \mathrm{vol}(\Omega) - \varepsilon N^{-\delta}$$



$$\sum_{k=1}^{N} \lambda_k \geq N^{1+\frac{2}{d}} \left(L^{\mathrm{cl}}_{0,d} \mathrm{vol}(\Omega) \right)^{-\frac{2}{d}} \frac{d}{2+d} - N^{2-\delta} A(\epsilon, \delta)$$

For $\delta < 1$ this improves the Melas bound.

Fig. 1. Minimizers of the functional $\int_{\mathbb{R}^2} |\xi|^2 F^*(|\xi|)\, d\xi$

Li-Yau, Melas and beyond

Put $d = 2$ and $V = \mathrm{vol}(\Omega)$. Moreover, let $J = \min_{a \in \mathbb{R}^d} \int_\Omega |x-a|^2 dx$.

Then we have the Li-Yau bound

$$\sum_{j=1}^{N} \lambda_j \geq \frac{2\pi}{V} N^2$$

and the Melas improvement

$$\sum_{j=1}^{N} \lambda_j \geq \frac{2\pi}{V} N^2 + \frac{V}{32\, J} N$$

Let Ω be a polygon with n sides.

Moreover, l_j is the length of the j-th side p_j and d_j is the distance of the middle third of p_j to $\partial\Omega \backslash p_j$.

Kovarik, Vougalter, W.

$$\sum_{k=1}^{N} \lambda_k \geq \frac{2\pi}{V} N^2 + \frac{4\alpha\, c_3}{V^{\frac{3}{2}}} N^{\frac{3}{2}-\epsilon(N)} \sum_{j=1}^{n} l_j\, \Theta\left(N - \frac{9V}{2\pi\, d_j^2} \right) + (1-\alpha) \frac{V}{32\, J} N$$

$$\sum_{k=1}^{N} \lambda_k = \frac{2\pi}{V} N^2 + \frac{16\sqrt{2\pi}}{9} \cdot \frac{|\partial\Omega|}{V^{\frac{3}{2}}} N^{\frac{3}{2}} + o(N^{\frac{3}{2}})$$

$$\epsilon(N) = \frac{2}{\sqrt{\log_2(2\pi N/c_1)}}, \qquad c_1 = \sqrt{\frac{3\pi}{14}}\, 10^{-11}, \qquad c_3 = \frac{2^{-3}}{9\sqrt{2}\, 36} (2\pi)^{\frac{5}{4}} c_1^{1/4}$$

Berezin and Li-Yau via Legendre transformation

For convex, non-negative $f(x)$, $x > 0$, the Legendre transformation is given by

$$f^\wedge(p) = \sup_{x>0}(px - f(x)), \quad p > 0\,.$$

From $f(x) \le g(x)$ it follows that $f^\wedge(p) \ge g^\wedge(p)$.

In view of

$$\Big(\sum_k (x-\lambda_k)_+\Big)^\wedge(p) = (p-[p])\lambda_{[p]+1} + \sum_{k=1}^{[p]} \lambda_k\,,$$

$$\left(L^{\rm cl}_{1,d}\mathrm{vol}(\Omega)x^{1+\frac{d}{2}}\right)^\wedge(p) = p^{1+\frac{2}{d}}\left(L^{\rm cl}_{0,d}\mathrm{vol}(\Omega)\right)^{-\frac{2}{d}}\frac{d}{2+d}.$$

Berezin`s bound

$$\sum_k (\Lambda-\lambda_k)_+ \le L^{\rm cl}_{1,d}\mathrm{vol}(\Omega)\Lambda^{1+\frac{d}{2}}$$

is dual to the Li-Yau inequality

$$\sum_{k=1}^N \lambda_k \ge N^{1+\frac{2}{d}}\left(L^{\rm cl}_{0,d}\mathrm{vol}(\Omega)\right)^{-\frac{2}{d}}\frac{d}{2+d}$$

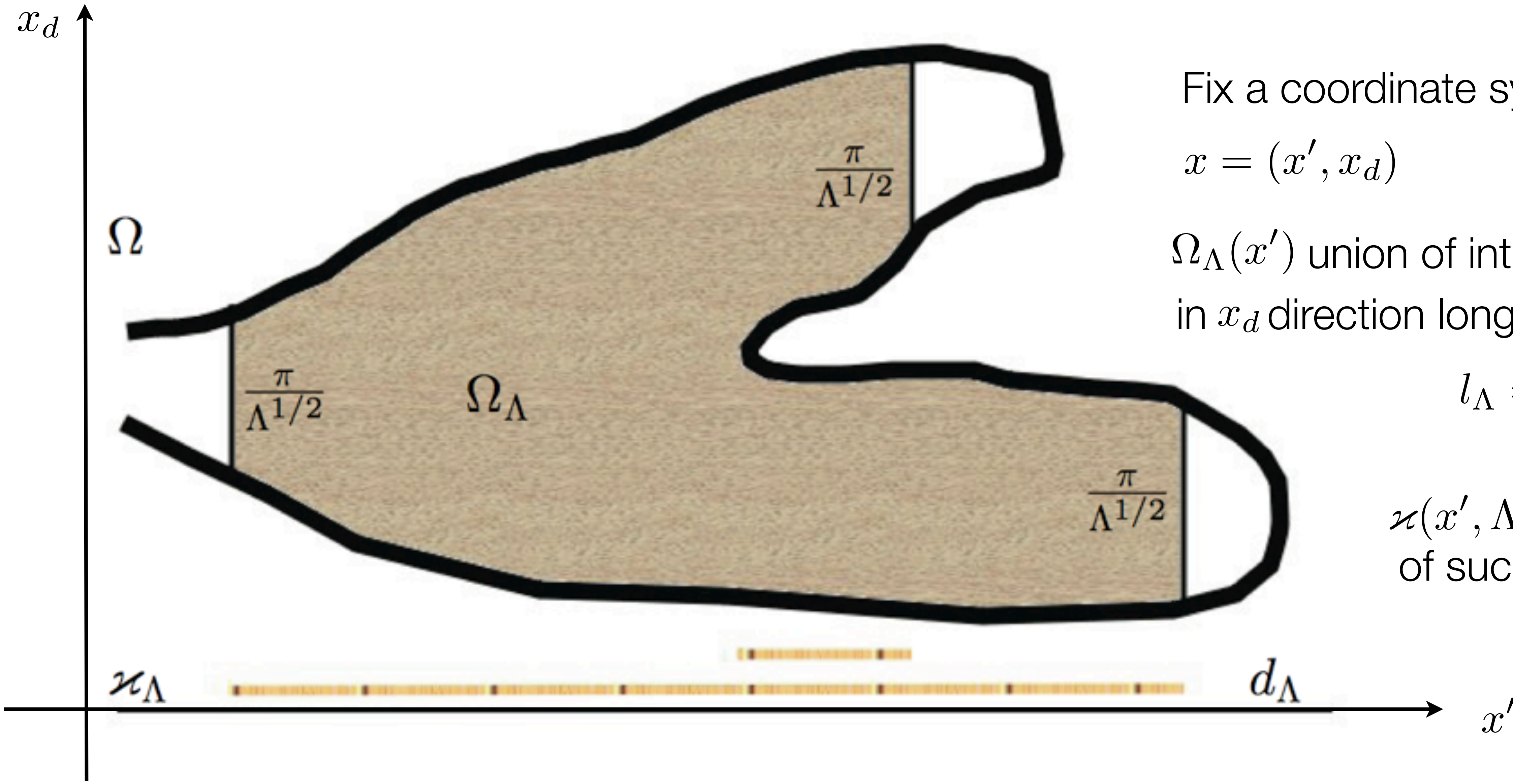
Reversely, the Melas improvement

$$\sum_{k=1}^N \lambda_k \ge N^{1+\frac{2}{d}}\left(L^{\rm cl}_{0,d}\mathrm{vol}(\Omega)\right)^{-\frac{2}{d}}\frac{d}{2+d} + M_d\frac{\mathrm{vol}(\Omega)}{J(\Omega)}\cdot N$$

is dual to a Berezin bound with energy shift

$$\sum_k (\Lambda-\lambda_k)_+ \le L^{\rm cl}_{1,d}\mathrm{vol}(\Omega)\left(\Lambda - M_d\frac{\mathrm{vol}(\Omega)}{J(\Omega)}\right)^{1+\frac{d}{2}}.$$

Improved Berezin bounds for $\sigma \geq 3/2$.



Fix a coordinate system.

$x = (x', x_d)$

$\Omega_\Lambda(x')$ union of intervals in Ω in x_d direction longer than

$$l_\Lambda = \frac{\pi}{\sqrt{\Lambda}}$$

$\varkappa(x', \Lambda)$ number of such intervals

$$\Omega_\Lambda = \bigcup_{x' \in \mathbb{R}^{d-1}} \Omega_\Lambda(x') \subset \Omega$$ subset of „local“ width exceeding $l_\Lambda = \frac{\pi}{\sqrt{\Lambda}}$

$$d_\Lambda(\Omega) = \int_{x' \in \mathbb{R}^{d-1}} \varkappa(x', \Lambda) dx'$$ „weighted“ projection of Ω_Λ

$\int \mathbf{Ad}\mu$

$S_{\sigma,d}(\Omega,\Lambda) \le S^{\rm cl}_{\sigma,d}(\Omega,\Lambda)$

Improved Berezin bounds for $\sigma \ge 3/2$.

For any $\Omega \subset \mathbb{R}^d$, $\sigma \ge 3/2$ and any $\Lambda > 0$ we have

$$S_{\sigma,d}(\Omega,\Lambda) \le \underbrace{L^{\rm cl}_{\sigma,d}\mathrm{vol}(\Omega_\Lambda)\Lambda^{\sigma+\frac{d}{2}}}_{S^{\rm cl}_{\sigma,d}(\Omega_\Lambda;\Lambda)} - \nu(\sigma,d)4^{-1}L^{\rm cl}_{\sigma,d-1}d_\Lambda(\Omega)\Lambda^{\sigma+\frac{d-1}{2}}$$

W.

The bound improves already the main term: It equals $S^{\rm cl}_{\sigma,d}(\Omega_\Lambda;\Lambda) \le S^{\rm cl}_{\sigma,d}(\Omega;\Lambda)$

Only the portion Ω_Λ of Ω which is wide enough enters the bound.

The bound contains also a negative term related to the second term in the Weyl asymptotics as $\Lambda \to +\infty$

$$S_{\sigma,d}(\Omega,\Lambda) = \underbrace{L^{cl}_{\sigma,d}\mathrm{vol}(\Omega)\Lambda^{\sigma+d/2}}_{S^{cl}_{\sigma,d}(\Omega,\Lambda)} - \frac{1}{4}L^{cl}_{\sigma,d-1}|\partial\Omega|\Lambda^{\sigma+(d-1)/2} + o(\Lambda^{\sigma+(d-1)/2})$$

The results remains true for the case of magnetic fields.

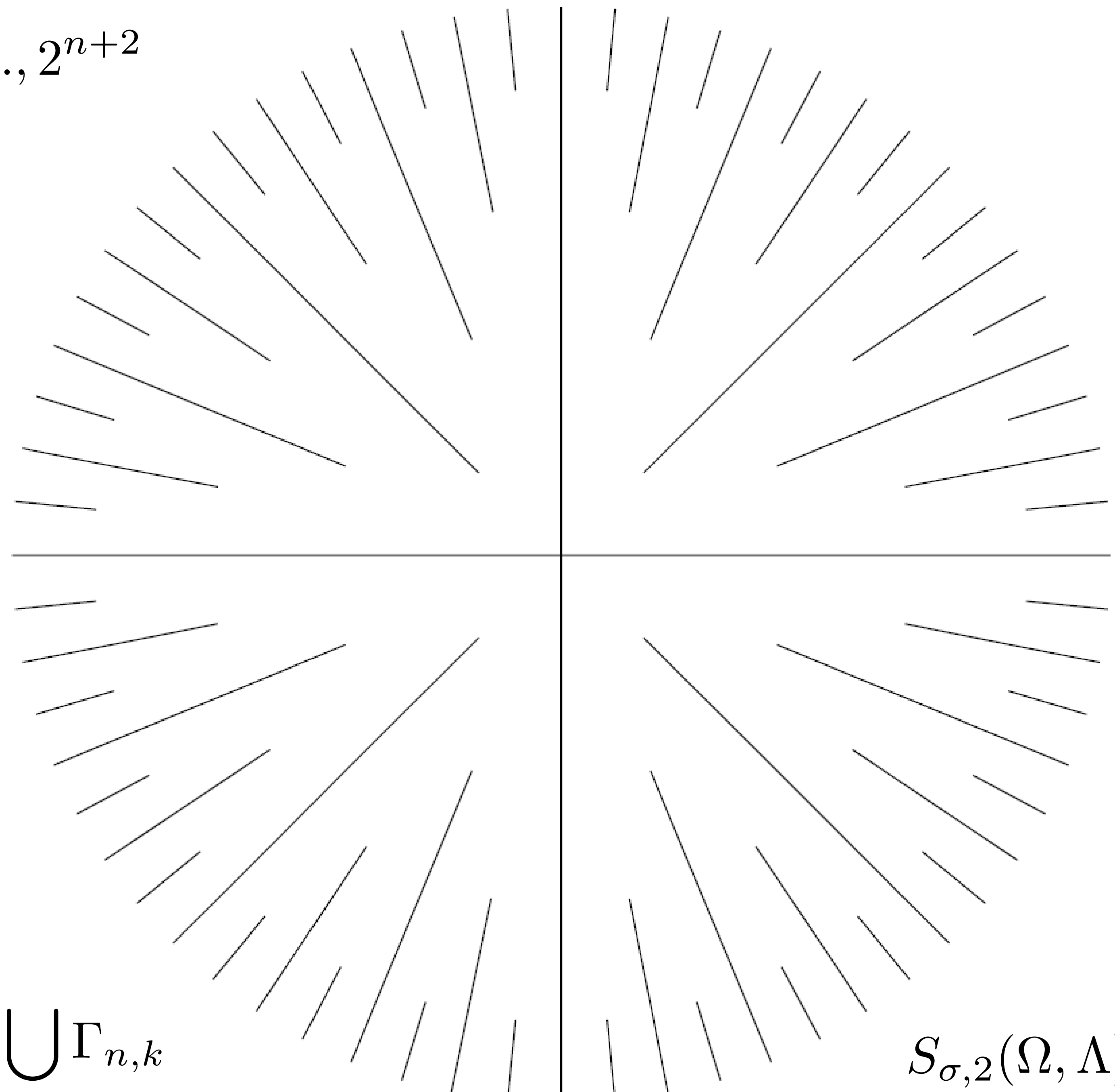
The results are applicable even to certain unbounded domains.

$\int \mathbf{A} d\mu$ $\quad \sigma \geq \frac{3}{2}$

Sharp Berezin bounds in unbounded domains

$$\Gamma_{k,n} = \left\{ (r,\phi) | r \geq r_n \, , \phi = \frac{k-1}{2^{n+1}} \pi \right\}$$

$$k = 1, ..., 2^{n+2}$$



$$\Omega = \mathbb{R}^2 \setminus \bigcup_{n,k} \Gamma_{n,k}$$

$$r_n = 2^{\delta n} \qquad 0 < \delta < 1$$

$$S_{\sigma,2}(\Omega, \Lambda) = 0$$

$$\Lambda < 15 \cdot 2^{-2(1+\delta)}$$

$$S_{\sigma,2}(\Omega, \Lambda) \leq C_{\sigma,\delta} \Lambda^{\sigma + \frac{1}{1-\delta}}$$

$$\Lambda \geq 15 \cdot 2^{-2(1+\delta)}$$

$$\lim_{n \to +\infty} 2^{-n} r_n = 0$$

$$r_n = n$$

$$S_{\sigma,2}(\Omega, \Lambda) = 0 \quad \text{if} \quad \Lambda < \frac{15}{4}$$

$$S_{\sigma,2}(\Omega, \Lambda) \leq C_\sigma \Lambda^{\sigma+1} (\ln \Lambda)^2 \quad \text{if} \quad \Lambda \geq \frac{15}{4}$$

Lieb-Thirring bounds for operator valued potentials

We consider Schrödinger operators with operator valued potentials

$$H = -\Delta \otimes \mathbf{1}_G - W(x) \quad \text{on} \quad L^2(\mathbb{R}^d, G)$$

Crucial is the following extention of the Lieb-Thirring bounds

$$\mathrm{tr}_{L^2(\mathbb{R}^d,G)}\; H_-^\sigma \le L_{\sigma,d}^{cl} \int \mathrm{tr}_G W_+^{\sigma+\frac{d}{2}}(x)dx, \quad \sigma \ge 3/2$$

Let us apply this to $-\Delta_\Omega^D$ for $d = 2$ and $\sigma = 3/2$.

By the variational principle we have

$$-\Delta_D^\Omega - \Lambda = -\frac{\partial^2}{\partial x_1^2} + \left(-\frac{\partial^2}{\partial x_2^2} - \Lambda\right) \ge -\frac{d^2}{dx_1^2} - W_-(x_1) \text{ on } L_2(\mathbb{R}, L_2(\mathbb{R}))$$

where $W(x_1) = (-d^2/dx_2^2)_D^{\Omega_\Lambda(x_1)} - \Lambda$ is the Dirichlet problem on the intervals in $\Omega_\Lambda(x_1)$

with the eigenvalues $\mu_k(x_1) \ge \pi^2 k^2 l^{-2}(x_1) - \Lambda,\; k \in \mathbb{N}.$ ($l(x_1)$ total length of $\Omega_\Lambda(x_1)$)

Closing the argument

$$S_{3/2,2}(\Omega,\Lambda) \quad \leq \quad S^{\rm cl}_{3/2,2}(\Omega_\Lambda,\Lambda) - \nu(3/2,2)\cdot 4^{-1} L^{cl}_{3/2,1}\cdot d_\Lambda(\Omega)\Lambda^2$$

$$S_{\frac{3}{2},2}(\Omega,\Lambda) \quad \leq \quad \operatorname{tr}\left(-\frac{d^2}{dx_1^2} - W_-(x_1)\right)_-^{3/2} \leq \frac{3}{16}\int_{\mathbb{R}} \operatorname{tr} W_-^2(x_1)dx_1$$

$$\leq \quad \frac{3}{16}\int_{\mathbb{R}}\sum_{k\geq 1}\left(\Lambda - \frac{\pi^2 k^2}{l^2(x_1)}\right)_+^2 dx_1 \leq \frac{3}{16}\int_{I_\Lambda}\frac{\pi^4}{l^4(x_1)}\sum_{k=1}^{[l(x_1)l_\Lambda^{-1}]}\left(\frac{l^2(x_1)}{l_\Lambda^2} - k^2\right)^2 dx_1$$

Here $I_\Lambda = \{x_1 | l(x_1) > l_\Lambda\}$. Estimating the sum as

$$\sum_{k=1}^{[l(x_1)l_\Lambda^{-1}]}\left(\frac{l^2(x_1)}{l_\Lambda^2} - k^2\right)^2 \leq \frac{8}{15}\frac{l^5(x_1)}{l_\Lambda^5} - \varepsilon(2)\frac{l^4(x_1)}{l_\Lambda^4} \quad \text{for} \quad l(x_1) > l_\Lambda$$

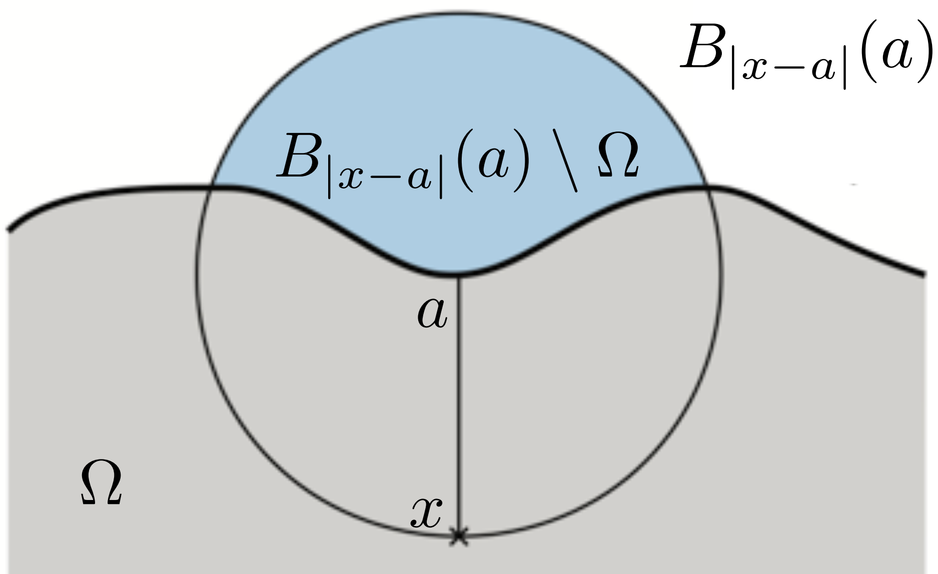
we finally get

$$S_{3/2,2}(\Omega,\Lambda) \quad \leq \quad \frac{3}{16}\cdot\frac{8}{15}\cdot\frac{1}{\pi}\cdot\Lambda^{5/2}\int_{I_\Lambda} l(x)dx - \frac{3}{16}\varepsilon(2)\Lambda^2\int_{I_\Lambda} dx$$

$$= \quad L^{cl}_{3/2,2}\mathrm{vol}(\Omega_\Lambda)\Lambda^{5/2} - \varepsilon(2)\cdot L^{cl}_{3/2,1}\cdot d_\Lambda\Lambda^2\,.$$

Here $0.475 < \varepsilon(2)$. The best possible constant in this spot can not exceed 0.5.

A more geometrical version of a two-term Berezin bound



$$\rho(x) = \frac{\mathrm{vol}(B_{|x-a|}(a) \setminus \Omega)}{\mathrm{vol}(B_{|x-a|}(a))}$$

$\delta(x)$ distance to the boundary

$$R(\Lambda) = \left\{ x \,|\, \delta(x) < 1/(4\sqrt{\Lambda}) \right\}$$

$$M_\Omega(\Lambda) = \int_{R(\Lambda)} \rho(x) dx$$

$$S_{\sigma,d}(\Omega,\Lambda) \le L^{cl}_{\sigma,d}\, \mathrm{vol}(\Omega) \Lambda^{\sigma+d/2} - L^{cl}_{\sigma,d}\, 2^{-d+1}\, \Lambda^{\sigma+d/2}\, M_\Omega(\Lambda)$$

Geisinger, Laptev, W.

The proof is based on the fact that for $W(x) = (-d^2/dx_2^2)_D^{\Omega_\Lambda(x_1)} - \Lambda$ we have

$$\mathrm{tr}\, W_-^\sigma(x_1) \le L^{\mathrm{cl}}_{\sigma,1} \int_{\Omega_\Lambda(x_1)} \left(\Lambda - \frac{1}{4\delta^2(t)} \right)_+^{\sigma+\frac{1}{2}} dt$$

Applications to the Heat kernel
Kac` inequality

If one uses the monotonicity of the (forward) Laplace transformation

$$\begin{aligned}\mathcal{L}[f](t) &= \int_0^\Lambda f(\Lambda)e^{-t\Lambda}d\Lambda\,,\\ \mathcal{L}[(\Lambda-\lambda)_+^\sigma](t) &= e^{-t\lambda}t^{-\sigma-1}\Gamma(\sigma+1), \quad t>0\,,\end{aligned}$$

then the Berezin bound for arbitrary $\sigma \geq 1$ implies

$$\begin{aligned}Z(t) = \mathrm{tr}\; e^{\Delta t} &= \sum_k e^{-\lambda_k t} = \sum_k \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\mathcal{L}[(\Lambda-\lambda_k)_+^\sigma](t)\\ &= \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\mathcal{L}[S_{\sigma,d}(\Omega,\Lambda)](t) \leq \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\mathcal{L}[S^{cl}_{\sigma,d}(\Omega,\Lambda)](t)\end{aligned}$$

and therefore

$$\begin{aligned}Z(t) &\leq \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}L^{cl}_{\sigma,d}\mathrm{vol}(\Omega)\mathcal{L}[\Lambda^{\sigma+d/2}](t)\\ &= \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\cdot\frac{\Gamma(\sigma+1)}{2^d\pi^{d/2}\Gamma(1+\sigma+d/2)}\cdot\mathrm{vol}(\Omega)\Gamma(1+\sigma+d/2)t^{-1-\sigma-d/2}\\ &= \frac{\mathrm{vol}(\Omega)}{(4\pi t)^{d/2}} \quad \text{for any} \quad t>0\,.\end{aligned}$$

An improvement of Kac` inequality

Using the Berezin version of the Melas improvement

$$S_{1,d}(\Omega,\Lambda) \le S^{\mathrm{cl}}_{1,d}\left(\Omega, \Lambda - M_d \frac{\mathrm{vol}(\Omega)}{J(\Omega)}\right)$$

Harrell and Hermi obtained

$$Z(t) \le \frac{\mathrm{vol}(\Omega)}{(4\pi t)^{d/2}} e^{-M_d \frac{\mathrm{vol}(\Omega)}{J(\Omega)} t}, \quad t > 0\,.$$

They conjectured

$$Z(t) \le \frac{\mathrm{vol}(\Omega)}{(4\pi t)^{d/2}} e^{-M_d \frac{t}{(\mathrm{vol}(\Omega))^{2/d}}}, \quad t > 0\,.$$

Put $\lambda \in [\tilde{\lambda}, \lambda_1]$ where $\tilde{\lambda} = \frac{\pi j^2_{\frac{d}{2}-1,1}}{(\Gamma(\frac{d}{2}+1)\mathrm{vol}(\Omega))^{2/d}} \le \lambda_1$ is the „Faber-Krahn eigenvalue“.

Then

$$Z(t) \;\le\; \frac{\mathrm{vol}(\Omega)}{(4\pi t)^{\frac{d}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d}{2} + 1, \lambda t\right) \;-\; (R(t,\lambda))_+$$

An improvement of Kac` inequality

In the bound

$$Z(t) \leq \frac{\mathrm{vol}(\Omega)}{(4\pi t)^{\frac{d}{2}}}\hat{\Gamma}\left(\sigma_d + \frac{d}{2} + 1, \lambda t\right) - (R(t,\lambda))_+$$

we use the normalized reduced Gamma function

$$\hat{\Gamma}(z,s) = \frac{1}{\Gamma(z)}\int_s^{+\infty} t^{z-1}e^{-t}dt$$

and the remainder term

$$R(t) = c_{1,d}\,\frac{(\mathrm{vol}(\Omega))^{\frac{d-1}{d}}}{(4\pi t)^{\frac{d-1}{2}}}\,\hat{\Gamma}\left(\sigma_d + \frac{d+1}{2}, \lambda t\right) - c_{2,d}\,\frac{(\mathrm{vol}(\Omega))^{\frac{d-3}{d}}}{(4\pi t)^{\frac{d-3}{2}}}\,\hat{\Gamma}\left(\sigma_d + \frac{d-1}{2}, \lambda t\right)$$

where

$$c_{1,d} = \frac{B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{2}\,\frac{\Gamma\left(\frac{d}{2}+1\right)^{\frac{d-1}{d}}}{\Gamma\left(\frac{d+1}{2}\right)} \quad \text{and} \quad c_{2,d} = c_{1,d}\,\frac{2\pi^2(d-1)\Gamma\left(\frac{d}{2}+1\right)^{-\frac{2}{d}}}{96(2\sigma_d + d - 1)}$$

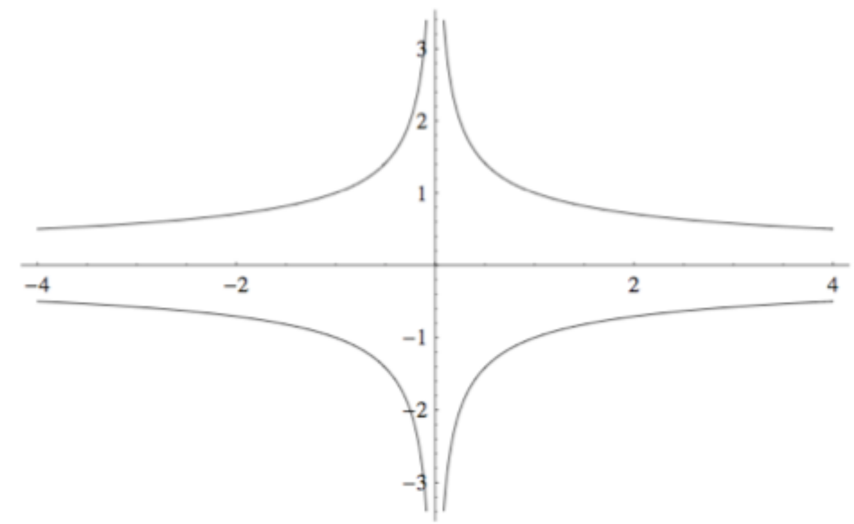
with $\sigma_2 = \dfrac{5}{2}, \sigma_3 = 2$ and $\sigma_d = \dfrac{3}{2}$ for $d \geq 4\,.$

This beats the conjecture up to $d = 633\,.$

Horn shaped domains

Consider two-dimensional horn shaped domains

$$\Omega_f = \left\{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}\,,\ |y| < f(x)\right\}.$$



Then for $f(s) = f_\mu(s) = s^{-\frac{1}{\mu}}$ with $\mu > 1$ one has

$$Z(t;\Omega_{f_\mu}) \quad = \quad \frac{\Gamma\left(1+\frac{\mu}{2}\right)\zeta(\mu)}{2\pi^{\mu+\frac{1}{2}}}\, t^{-\frac{\mu+1}{2}} + o\left(t^{-\frac{\mu+1}{2}}\right) \quad \text{as} \quad t \to 0+ \ .$$

Cutting the domain and dropping the remainder term we get in view of

$$Z(t) \quad = \quad \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\mathcal{L}[S_{\sigma,d}(\Omega,\Lambda)] \le \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\mathcal{L}[S^{cl}_{\sigma,d}(\Omega_\Lambda,\Lambda)]$$

the asymptotically sharp estimate

$$Z(t;\Omega_{f_\mu}) \le \frac{\Gamma\left(1+\frac{\mu}{2}\right)\zeta(\mu)}{2\pi^{\mu+\frac{1}{2}}}\, t^{-\frac{\mu+1}{2}} \quad \text{for all} \quad t > 0 \quad \text{and} \quad \mu > 1\,.$$

∫**A**dμ

Open Problems

1. Settle Pólya‘s hypothesis! $n_D^\Omega(\Lambda) = S_{0,d}(\Omega,\Lambda) \leq S^{\mathrm{cl}}_{0,d}(\Omega,\Lambda) = \frac{\tau_d}{(2\pi)^d}\Lambda^{\frac{d}{2}}\mathrm{vol}(\Omega)$

2. Improve the Berezin bound for $\sigma = 1$ in any of the following ways:

$$
\begin{aligned}
S_{1,d}(\Omega,\Lambda) &\leq S^{cl}_{1,d}(\Omega_\Lambda,\Lambda)\\
S_{1,d}(\Omega,\Lambda) &\leq S^{cl}_{1,d}(\Omega,\Lambda) - c(d)d_\Lambda(\Omega)\Lambda^{1+\frac{d-1}{2}}\\
S_{1,d}(\Omega,\Lambda) &\leq S^{cl}_{1,d}(\Omega_\Lambda,\Lambda) - c(d)d_\Lambda(\Omega)\Lambda^{1+\frac{d-1}{2}}\\
S_{1,d}(\Omega,\Lambda) &\leq S^{cl}_{1,d}(\Omega,\Lambda) - c(d)\Lambda^{1+\frac{d}{2}}M_\Omega(\Lambda)
\end{aligned}
$$

Thank you for your attention!