Spectral Methods in PDE

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I. Introduction

We develop a new geometric L^2-method for nonlinear PDE. I will introduce this method in the NLS and NLW contexts and conclude with some comments on the Euler equation.

II. NLS on $\mathbb{T}^d$

We study NLS on the torus in arbitrary dimensions d:

$$-i\dot{u} = -\Delta u + |u|^{2p}u + H(x,u), \quad p \in \mathbb{N} \text{ arbitrary},$$

where H is analytic, has a higher order zero at zero and could have x dependence, which break translation invariance. Since the method is indifferent to the lack of symmetry and is stable under perturbation, we take $H = 0$ for this talk.

Remark. Stability comes from the fact that this is an L^2-method. More comments later on connections with geometry and (lack of) integrability.

We recall that the local theory (existence and uniqueness in a time interval of length $\mathcal{O}(1)$) for NLS is in H^s for

$$s > \max(0, \frac{1}{2}(d - \frac{2}{p}))$$

from L^{2p+2} (Strichartz) estimates [Bourgain], which linearizes about the flow of the Laplacian.

In higher dimensions, $s > 1$ for p which are not very large. E. g., $d = 4$, the local theory for the quintic NLS is in $H^{3/2}$, where there is no available conservation law and is therefore energy supercritical. The main difficulty here is that the linear flow cannot be used as the reference flow as in the usual Duhamel formula.

The existence of global flow is generally unknown because of this lack of a conservation law and the additional lack of dispersion on the torus. The result that I present below constructs a class of global solutions to NLS and NLW on the torus in arbitrary dimensions.

Remark. This class of global flow in the energy supercritical regime is in some sense "non-classical". In the subcritical (or critical) regime the flow could be viewed as "classical". More on this later.

The method that I will present analyzes the geometry of the nonlinearity relative to the bi-characteristics and does not make use of conservation laws. A consequence is that it applies to both focusing and defocusing cases and has particular relevance in the energy supercritical context. In fact, the method relies little on the specifics of the equation and appears to be general.

We note that this geometric analysis here is necessitated by the complete violation of the Kolmogorov non-degeneracy conditions or its weaker versions (on the Hamiltonian).

From the oscillatory integral point of view, it is about choosing a phase other than the Laplacian.

III. The Non-Linear Space - Time Fourier Series

The solutions to the linear Schrödinger equation on the d-torus:

$$-i\dot{u} = -\Delta u,$$

are provided by spectral theory. They are linear combinations of the eigenfunction solutions:

$$u = e^{-i\omega_j t} e^{ij\cdot x}.$$

Since $\omega_j = j^2$, e. v. of the Laplacian, they are time periodic.

As an ansatz, we seek solutions of b frequencies to the nonlinear equation

$$-i\dot{u} = -\Delta u + |u|^{2p} u$$

in the form of a nonlinear Fourier series:

$$u = \sum \hat{u}(n, j) e^{in\cdot\omega t} e^{ij\cdot x}, \quad (n, j) \in \mathbb{Z}^{b+d},$$

where $\omega \in \mathbb{R}^b$ is to be determined. We note that for each frequency in time we add a dimension.

Remark. In the usual restriction theorem, we add one dimension for time.

This is the so called amplitude-frequency modulation, fundamental to nonlinear equations. For linear equations, ω are the eigenvalues, they are fixed once and for all. In this language, a solution $u^{(0)}$ to the linear equation can be written as

$$u^{(0)} = \sum_j \hat{u}(e_j, j) e^{-i\omega_j^{(0)} t} e^{ij\cdot x}, \quad (n, j) \in \mathbb{Z}^{b+d},$$

where e_j is a basis vector in $\mathbb{Z}^b$ and $\omega_j^{(0)} = e_j \cdot \omega^{(0)} = j^2$.

If we succeed in their constructions, these quasi-periodic solutions play a role related to "eigenfunctions" in the nonlinear setting by providing a basis for the (smooth) flow for NLS.

Remark. For NLW, reflecting the dense nature of the spectrum of $\sqrt{-\Delta}$ in dimensions two and above, this is not enough. The higher part of the spectrum just does not have a similar structure as the lower one. The geometry is also more complicated as we will see.

IV. The Bi-Characteristics

In the Fourier space $\mathbb{Z}^{b+d}$, the support of the solution in the above form to the linear equation and its complex conjugate are by definition, the bi-characteristics $\mathcal{C}$:

$$\mathcal{C} = \{(n, j) \in \mathbb{Z}^{b+d} | \pm n \cdot \omega^{(0)} + j^2 = 0\}.$$

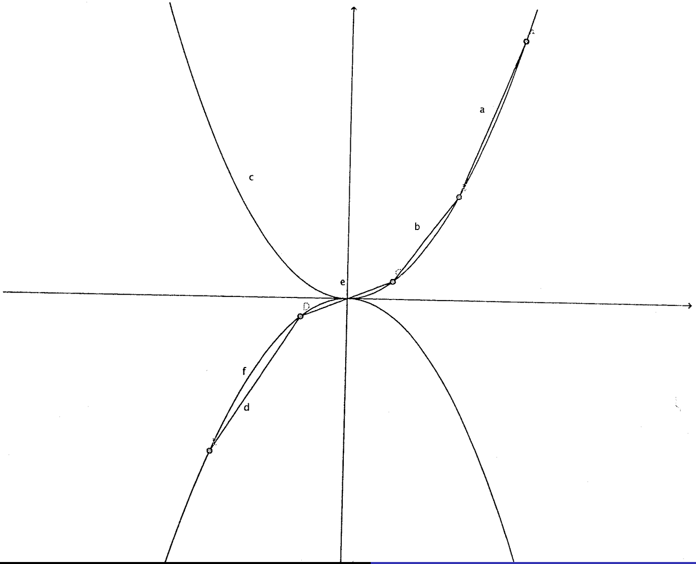
We further define

$$\begin{aligned} \mathcal{C}^+ &= \{(n, j) | n \cdot \omega^{(0)} + j^2 = 0\} \\ \mathcal{C}^- &= \{(n, j) | - n \cdot \omega^{(0)} + j^2 = 0\}. \end{aligned}$$

It is convenient to consider these as subsets of two copies of $\mathbb{Z}^{b+d}$ (the usual doubling of dimensions) and therefore we have

$$\mathcal{C} = \mathcal{C}^+ \times \mathcal{C}^- \subset \mathbb{Z}^{b+d} \times \mathbb{Z}^{b+d}.$$

1d periodic for NLS:



$\mathcal{C}$ is the support of the solution to the linear equation. We consider $\mathcal{C}$ as the restriction to $\mathbb{Z}^{b+d} \times \mathbb{Z}^{b+d}$ of the corresponding manifold, the paraboloids on $\mathbb{R}^{b+d} \times \mathbb{R}^{b+d}$. So $\mathcal{C}$ is a manifold of singularities and not just isolated points. Moreover since $\omega^{(0)}$ is an integer vector, $\mathcal{C}$ not only lacks convexity but also has null directions in n.

Using the ansatz, we use a Newton scheme to solve the nonlinear equation. To start this scheme we need a spectral gap (invertibility of a linearized operator). This spectral gap will come from analyzing the geometry of the bi-characteristics.

The main novelty is that this gap is created by the nonlinearity itself and not from the linear operator. It is non-perturbative leading to a high frequency semi-classical theory as well. It is non-perturbative, because it comes from geometry. Recall that the geometry is non-convex here.

Remark. In infinite dimensions, another natural parameter is $1/N$ where N is the finite dimension approximation. This is the high frequency or semi-classical parameter.

V. The Nonlinear Matrix Equation and Newton Scheme

Using the ansatz, the nonlinear equation becomes:

$$\text{diag}\,(n\cdot\omega + j^2)\hat{u} + (\hat{u} * \hat{v})^{*p} * \hat{u} = 0$$

where $(n, j) \in \mathbb{Z}^{b+d}$, $\hat{v} = \hat{\bar{u}}$ and $\omega \in \mathbb{R}^b$ is to be determined. For simplicity we drop the hat and write u for $\hat{u}$ and v for $\hat{v}$ etc.

We seek solutions close to the linear solution $u^{(0)}$ of b frequencies, supp $u^{(0)} = \{(-e_k, j_k), k = 1, ..., b\}$, with frequencies $\omega^{(0)} = \{j_k^2\}_{k=1}^b$ $(j_k \neq 0)$.

We complete the above equation by writing the equation for the complex conjugate. So we have the system:

$$\text{diag } (n \cdot \omega + j^2) u + (u * v)^{*p} * u = 0,$$

$$\text{diag } (-n \cdot \omega + j^2) v + (u * v)^{*p} * v = 0.$$

When the nonlinearity is absent, i.e., $p = 0$, the above system gives the bi-characteristics, the paraboloids $\mathcal{C}$, which is the eigen-space of the eigenvalue 0. This is a set of co-dimension 1, an infinite set. We are therefore in a setting which is non- elliptic (or non sub-elliptic).

We use a Newton scheme to solve the above equations, with $u^{(0)}$ as the initial approximation. We recall the formal Newton scheme: the first correction

$$\Delta \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix} = \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix} - \begin{pmatrix} u^{(0)} \\ v^{(0)} \end{pmatrix} = [F'(u^{(0)}, v^{(0)}]^{-1} F(u^{(0)}, v^{(0)}),$$

where $\begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}$ is the next approximation and $F'(u^{(0)}, v^{(0)})$ is the linearized operator on $\ell^2(\mathbb{Z}^{b+d}) \times \ell^2(\mathbb{Z}^{b+d})$.

The convergence is double exponential:

$$|F(u + \Delta u)| = \mathcal{O}(|F(u)|^2),$$

provided one has adequate control over the linearized operator.

VI. Generic Linear Solutions and Invertibility of $F'(u^{(0)})$

Concretely we have

$$F' = D + A,$$

where

$$D = \begin{pmatrix} \text{diag}\,(n\cdot\omega + j^2) & 0 \\ 0 & \text{diag}\,(-n\cdot\omega + j^2) \end{pmatrix}$$

and

$$A = \begin{pmatrix} (p+1)(u*v)^{*p} & p(u*v)^{*p-1}*u*u \\ p(u*v)^{*p-1}*v*v & (p+1)(u*v)^{*p} \end{pmatrix}$$

with $\omega = \omega^{(0)}$, $u = u^{(0)}$ and $v = v^{(0)}$.

Since we look at small data, $\|A\| = \mathcal{O}(\delta^{2p}) \ll 1$ and the diagonal: $\pm n\cdot\omega + j^2$ are integer valued, using the Schur complement reduction, the spectrum of F' around 0 is equivalent to that of a reduced operator on $\ell^2(\mathcal{C})$.

So it suffices to establish a spectral gap for the convolution operator A restricted to the paraboloids $\mathcal{C}$. The difficulty here is that $\mathcal{C}$ is not convex because it is flat in the n-directions, compatible with convolution.

However as we show below, for generic linear solutions, the size of the sets on $\mathcal{C}$ connected by the convolution operator A is uniformly bounded. So the matrix A is a direct sum of finite matrices and hence 0 is typically not in the spectrum leading to a spectral gap.

The genericity conditions come from bounding the size of connected sets on $\mathcal{C}$. This can be seen as follows. Assume $(n, j) \in \mathcal{C}^+$ is connected to $(n', j') \in \mathcal{C}$ by the convolution operator A, then $n' = n + \Delta n$ and $j' = j + \Delta j$, where $(\Delta n, \Delta j) \in$ supp $(u^{(0)} * v^{(0)})^{*p}$, if $(n', j') \in \mathcal{C}^+$ and

$$(n \cdot \omega^{(0)} + j^2) = 0,$$

$$(n + \Delta n) \cdot \omega^{(0)} + (j + \Delta j)^2 = 0;$$

and if $(n', j') \in \mathcal{C}^-$, then $(\Delta n, \Delta j) \in$ supp $(u^{(0)} * v^{(0)})^{*p-1} * u^{(0)} * u^{(0)}$ and

$$(n \cdot \omega^{(0)} + j^2) = 0,$$

$$-(n + \Delta n) \cdot \omega^{(0)} + (j + \Delta j)^2 = 0.$$

(Clearly the situation is similar if $(n, j) \in \mathcal{C}^-$.)

The above defines a system of linear equations. By requiring the determinant (resultant) to be non-zero, we obtain that aside from the degenerate case, the connected sets are of sizes at most $d+2$. The degenerate case results from translation invariance and has spatial support in the set $\{\pm j_k\}_{k=1}^{b}$.

The resultant being non-zero gives the main part of the genericity conditions. For generic linear solutions, $A = \oplus_\alpha A_\alpha$.

The determinant of A_α is a polynomial in the Fourier coefficients $a = \{a_k\}_{k=1}^{b}$. It is easy to show that this is not a constant polynomial by restricting to $a = \{a_1, 0, ..., 0\}$. Requiring the determinant to be non-zero then gives the spectral gap on a set of Fourier coefficients a of positive measure.

Let P be the projection onto the paraboloids $\mathcal{C}$. Invertibility of F' then follows from Schur complement reduction:

$\lambda \in \sigma(F')$ if and only if $0 \in \sigma(H)$, where

$$H = PF'P - \lambda + PF'P^c(P^cF'P^c - \lambda)^{-1}P^cF'P,$$

by taking $\lambda = 0$.

So we have the following lemma, which is at the basis of this construction:

The Spectral Gap Lemma. For generic linear solutions, the linearized operator F' has a spectral gap on a set of Fourier coefficients of positive measure. The non-generic set is of co-dimension 1 given by algebraic equations.

Remark. This lemma continues to hold in the high-frequency semi-classical case by considering the bi-characteristics fixed also at integers other than 0.
In other words, for high- frequencies, the bi-characteristic are approximate invariants and used to fibre the dual space $\mathbb{Z}^{b+d}$, the Fourier space.

The above spectral gap lemma leads, for example, to the following results:

VII. Theorems

Theorem 1. For any b, there exists a set $\Omega \subset (\mathbb{R}^d)^b$ of codimension 1. Assume $j = \{j_k\}_{k=1}^b \in (\mathbb{R}^d)^b \backslash \Omega$ and $u^{(0)} = \sum_{k=1}^b a_k e^{-ij_k^2 t} e^{ij_k \cdot x}$ a solution to the linear equation with b frequencies and $a = \{a_k\} \in (0, \delta]^b$. There exist C, $c > 0$, such that for all $\epsilon \in (0, 1)$, there exists $\delta_0 > 0$ and for all $\delta \in (0, \delta_0)$ a Cantor set $\mathcal{G}$ with

$$\text{meas } \{\mathcal{G} \cap \mathcal{B}(0, \delta)\} / \delta^b \geq 1 - C\epsilon^c.$$

For all $a \in \mathcal{G}$, there is a quasi-periodic solution of b frequencies to the nonlinear Schrödinger equation

$$u(t, x) = \sum a_k e^{-i\omega_k t} e^{ij_k \cdot x} + \mathcal{O}(\delta^3),$$

with basic frequencies $\omega = \{\omega_k\}$ satisfying

$$\omega_k = j_k^2 + \mathcal{O}(\delta^{2p}).$$

The remainder $\mathcal{O}(\delta^3)$ is in an appropriate analytic norm on $\mathbb{T}^{b+d}$.

Theorem 2. Let $u_0 = u_1 + u_2$. Assume u_1 is generic and $\|u_2\| = \mathcal{O}(\delta)$, where $\|\cdot\|$ is an analytic norm on $\mathbb{T}^d$. Then for all $A > 1$, there exists an open set $\mathcal{A} \subset \mathcal{B}(0,1)$ of positive measure and $\delta_0 > 0$, such that for all $\delta \in (0, \delta_0)$, if $\{|\hat{u}_1|\} \in \mathcal{A}$, then (*) has a unique solution $u(t)$ for $|t| \le \delta^{-A}$ satisfying $u(t=0) = u_0$ and $\|u(t)\| \le \|u_0\| + \mathcal{O}(\delta)$. Moreover if $u_2 = 0$, *meas* $\mathcal{A} \to 1$ as $\delta \to 0$.

Corollary 1. Assume

$$u^{(0)}(t,x)=\sum_{k=1}^{b} a_k e^{-ij_k^2 t} e^{ij_k\cdot x},$$

a solution to the linear equation is generic with $\{j_k\}_{k=1}^{b}\in[K\mathbb{Z}^d]^b$, $K\in\mathbb{N}^+$ and $a=\{a_k\}\in(0,1]^b=\mathcal{B}(0,1)$. There exist C, $c>0$, such that for all $\epsilon\in(0,1)$, there exists $K_0>0$ and for all $K>K_0$ a Cantor set $\mathcal{G}$ with

$$\text{meas}\,\{\mathcal{G}\cap\mathcal{B}(0,1)\}\geq 1-C\epsilon^c.$$

For all $a \in \mathcal{G}$, there is a quasi-periodic solution of b frequencies to the nonlinear Schrödinger equation:

$$u(t, x) = \sum a_k e^{-i\omega_k t} e^{ij_k \cdot x} + \mathcal{O}(1/K^2),$$

with basic frequencies $\omega = \{\omega_k\}$ satisfying

$$\omega_k = j_k^2 + \mathcal{O}(1).$$

The remainder $\mathcal{O}(1/K^2)$ is in an analytic norm about a strip of width $\mathcal{O}(1)$ in t and $\mathcal{O}(1/K)$ in x on $\mathbb{T}^{b+d}$.

Corollary 2. Assume u_0 is *generic* with frequencies $\{j_k\}_{k=1}^b \in [K\mathbb{Z}^d]^b$, $K \in \mathbb{N}^+$. Let $\mathcal{B}(0,1) = (0,1]^b$. Then for all $A > 1$, there exist an open set $\mathcal{A} \subset \mathcal{B}(0,1)$ of positive measure and $K_0 > 0$, such that for all $K > K_0$, if $\{|\hat{u}_0|\} \in \mathcal{A}$, then the nonlinear Schrödinger equation has a unique solution $u(t)$ for $|t| \le K^A$ satisfying $u(t=0) = u_0$ and $\|u(t)\| \le \|u_0\| + \mathcal{O}(1/K^2)$, where $\|\cdot\|$ is an analytic norm (about a strip of width $\mathcal{O}(1/K)$) on $\mathbb{T}^d$, moreover meas $\mathcal{A} \to 1$ as $K \to \infty$.

Remark. The previous related results [Carles] are only to time $\mathcal{O}(K)$ by solving the associated Hamilton-Jacobi equations, before the arrival of caustics. Here we put space and time on equal footing and take a space-time approach.

We note that these seem to be the first instances where global solutions are constructed for energy supercritical NLS. We use geometry and spectral methods to bypass the lack of Sobolev embedding into a useful space (usually H^1) with an available conservation law.

In some sense, the new method is about how to deal with first order operators, namely the operator:

$$i\frac{\partial}{\partial t},$$

and its entailed "non-classical" flow.

IX. A Word on the Constrution of Quasi-Periodic Solutions

The spectral gap lemma provides the good geometry, which is essential to start the analysis. The ensuing analysis adapts the iteration scheme of Bourgain. The adaptation is needed because this is singular perturbation theory.

We note that the Bourgain scheme has as its predecessor the Craig-Wayne scheme, which is in turn inspired by the Fröhlich-Spencer Anderson localization.

X. Remarks on the Method

I would like to insert this method into two other general contexts (algebraic and geometric) .

1. Preparation theorem (polynomial)

Using the geometric incompatibility of the nonlinearity with the bi-characteristics for generic linear solutions, the linearized operator, which is an infinite matrix decomposes into a direct sum of finite matrices. So the determinant becomes the product of polynomials of bounded degree and moreover there are only finite types of such polynomials. This enables us to show that generically the determinant is non-zero.

2. Sobolev embedding

When one finds an appropriate space of functions, Sobolev embedding is available for all functions in the space. In the present setting this is no longer the case, the p in the L^p nonlinearity is too large and there are geometric obstructions.

In some sense the method shows that under appropriate conditions, this obstruction pertains only to a set of co-dimenison 1. So away from it, one could continue the analysis and construct a global flow.

XI. NLW

We consider *real* valued solutions to the nonlinear wave equation on the d-torus $\mathbb{T}^d = [0, 2\pi)^d$:

$$\frac{\partial^2 v}{\partial t^2} - \Delta v + v + v^{p+1} = 0 \qquad (p \geq 1, p \in \mathbb{N} \text{ arbitrary}),$$

with periodic boundary conditions: $v(t, x) = v(t, x + 2j\pi)$, $x \in [0, 2\pi)^d$ for all $j \in \mathbb{Z}^d$ and $v \in \mathbb{R}$. We use the standard ODE technique to write the above equation as a first order equation in t.

Let

$$D = \sqrt{-\Delta + 1}$$

and

$$u = (v, D^{-1}\frac{\partial v}{\partial t}) \in \mathbb{R}^2.$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we then obtain the corresponding first order equation

$$i\frac{\partial u}{\partial t} = Du + D^{-1}(\frac{u + \bar{u}}{2})^{p+1}.$$

We note that the bi-characteristics for the wave equation are hyperbolic, being the hyperboloids. As far as stability issues are concerned, this presents a more difficult geometry. The analysis difficulty comes from that fact that the spectrum of $D = \sqrt{-\Delta + 1}$ is not close to a discrete set and is dense in dimensions two and above.

Remark. Contrary to the linear Schrödinger, whose solutions are all periodic; the solutions to the linear wave equation are in general quasi-periodic. This is in fact the generic case on compact manifolds, cf. [Duistermaat, Guillemin].

So NLW puts this spectral gap method in its general form both from the geometric and the analysis angles.

XII. Remarks on the Euler Vortex Patches

This method has possible applications to the geometric stability of Kirchhoff ellipses (1876). More precisely one looks for divergence free solutions of the (incompressible) Euler equation in the Euclidean plane:

$$\dot{u} + u \cdot \nabla u + \nabla p = 0$$

with initial condition which is the characteristic function of a bounded open set – a patch.

The Euler equation is an area preserving transformation. The question regards lower dimensional quantities such as the circumference of the patch, which is not a conserved quantity.

This method could possibly address L^∞ geometric stability for data appropriately close to the Kirchhoff ellipses with small eccentricity. In particular this should exclude "fingering".

Previously it is known that the boundary of a smooth patch remains smooth for all time [Chemin, 1993]. But two smooth curves enclosing the same area could have no relation with each other.

It is also known that there is L^1 Lyapunov stability [Wan, Pulvirenti, 1985]. But this does not imply geometric stability of the boundary in any metric.

One should note that these vortex patches are only piece-wise smooth data with jump discontinuity across sets of co-dimension 1. The global existence and uniqueness of such weak solutions in $L^\infty \cap L^1$ is established by Yudovich.

The method of Arnold, cf. [Ebin, Marsden, 1970], which maps onto the manifold of groups of volume preserving diffeomorphisms is in the C^∞ category and not applicable. Instead we map onto a problem in Euclidean geometry and establish a spectral gap. We note that the fact that our method does not use the Hamiltonian structure becomes essential here.

XIII. Conclusion

To conclude, the main theme of the talk is the construction of a class of global non-classical, nonlinear flow by developing methods based on (classical) algebraic geometry and Fourier series analysis.

XIV. Some Literature on NLS and NLW

1. NLS
With external parameter (spectrally defined Laplacian): Bourgain, Eliasson-Kuksin, Geng-You,

2. NLW
With external parameter (spectrally defined square root of Laplacian): Bourgain