Weyl laws for non-self-adjoint differential operators with small random perturbations

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Spectral days, Munich, April 10–13, 2012

soutenu par une bourse FABER Bourgogne Conseil régional et par les contrats JC05-52556, ANR-08-BLAN-0228-01 de l'ANR.

0. Introduction

Non-self-adjoint spectral problems appear naturally e.g.:

- Resonances, (scattering poles) for self-adjoint operators, like the Schrödinger operator,
- The Kramers–Fokker–Planck operator

$$y \cdot h\partial_x - V'(x) \cdot h\partial_y + \frac{\gamma}{2}(y - h\partial_y) \cdot (y + h\partial_y).$$

A major difficulty is that the resolvent may be very large even when the spectral parameter is far from the spectrum:

$$\|(z - P)^{-1}\| \gg \frac{1}{\operatorname{dist}(z, \sigma(P))},$$

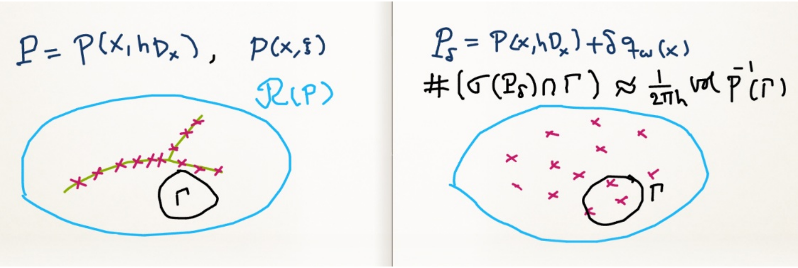
$\sigma(P) =$ spectrum of P. This implies that $\sigma(P)$ is unstable under small perturbations of the operator. (Here $P : \mathcal{H} \to \mathcal{H}$ is a closed operator and $\mathcal{H}$ a complex Hilbert space.)

In the case of (pseudo-)differential operators this follows from the Hörmander–Davies quasimode construction [Ho60a, Ho60b], [Da99], valid when the Poisson bracket of the symbol and it complex adjoint is $\neq 0$. Related problems:

- Numerical instability,
- No spectral resolution theorem in general,
- Difficult to study the distribution of eigenvalues.
- Significance of eigenvalues for instance in evolution problems?

Pleasant fact: The spectral instability also leads to very general results about Weyl asymptotics of eigenvalues of differential operators when adding a small random perturbation. See Hager [Ha05, Ha06a, Ha06b, HaSj08] , W. Bordeaux Montrieux [Bo08, Bo11, BoSj10] and the speaker [Sj08a, Sj08b] showing that for large classes of elliptic (pseudo-)differential operators with small random perturbations, with probability close or equal to 1, the eigenvalues distribute according to the Weyl law, wellknown in the self-adjoint case.
Also related works by Hager-Davies [DaHa09] and T. Christinsen-Zworski [ChrZw09].

This was quite surprising, since eigenvalues of ordinary differential operators with analytic coefficients tend to obey a complex Bohr-Sommerfeld quantization condition and are confined to a union of curves.



Plan of this talk:

1) Discussion of an earlier result about Weyl asymptotics of eigenvalues for elliptic semi-classical operators with a simplified presentation.
2) Presentation of a recent result about Weyl asymptotics for resonances near the real axis for Schrödinger operators.

1. Eigenvalues of elliptic semi-classical operators

The original 1D result of Hager was generalized in many ways and we present here (in a simplified formulation) a general result for elliptic semi-classical operators on compact manifolds with a small random perturbation in the 0 order term.

Let X be a compact n-dimensional manifold. The unperturbed operator is

$$P = \sum_{|\alpha| \leq m} a_\alpha(x; h)(hD)^\alpha. \tag{1}$$

The perturbed operator is

$$P + W, \; W = W(x) \text{ is the random perturbation.} \tag{2}$$

Assume

$$a_\alpha(x;h) = a_\alpha^0(x) + \mathcal{O}(h) \text{ in } C^\infty, \tag{3}$$
$$a_\alpha(x;h) = a_\alpha(x) \text{ is independent of } h \text{ for } |\alpha| = m.$$

Let

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \tag{4}$$

Assume that P is elliptic,

$$|p_m(x,\xi)| \geq \frac{1}{C}|\xi|^m, \tag{5}$$

and that $p_m(T^*X) \neq \mathbf{C}$.

Let $p = \sum_{|\alpha|\le m} a_\alpha^0(x)\xi^\alpha$ be the semi-classical principal symbol. We make the symmetry assumption

$$P = \Gamma P^* \Gamma =: P^{\mathrm{t}}, \tag{6}$$

where P^* denotes the complex adjoint with respect to some fixed smooth positive density of integration and Γ is the antilinear operator of complex conjugation; $\Gamma u = \overline{u}$. Notice that this assumption implies that

$$p(x, -\xi) = p(x, \xi). \tag{7}$$

Let $V_z(t) := \mathrm{vol}\left(\{\rho \in T^*X;\ |p(\rho) - z|^2 \le t\}\right)$. For $\kappa \in]0, 1]$, $z \in \mathbf{C}$, we consider the non-flatness property that

$$V_z(t) = \mathcal{O}(t^\kappa),\ 0 \le t \ll 1. \tag{8}$$

We see that (8) holds with $\kappa = 1/(2m)$.

Theorem ([Sj08b])

Let $\Gamma \Subset \mathbf{C}$ have smooth boundary, let $\kappa \in]0,1]$ and assume that (8) holds uniformly for z in a neighborhood of $\partial\Gamma$. Then there exists a probability measure supported on $\cap_{\widetilde{s},N\geq 0} B_{H^{\widetilde{s}}}(0, C_{\widetilde{s},N} h^N)$ and constants $C, N_0 > 0$ such that for $C^{-1} \geq r > 0$, $\widetilde{\epsilon} \geq C\epsilon_0(h)$, $\epsilon_0(h) := h^\kappa (\ln 1/h)^3$ we have with probability (for W)

$$\geq 1 - \frac{C}{rh^{N_0}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} \tag{9}$$

that:

$$|\#(\sigma(P+W)\cap\Gamma) - \frac{1}{(2\pi h)^n}\mathrm{vol}\,(p^{-1}(\Gamma))| \leq \tag{10}$$
$$\frac{C}{h^n}\left(\frac{\widetilde{\epsilon}}{r} + C\left(r + \ln(\frac{1}{r})\mathrm{vol}\,\left(p^{-1}(\partial\Gamma + D(0,r))\right)\right)\right).$$

Here $\#(\sigma(P+W)\cap\Gamma)$ denotes the number of eigenvalues of $P+W$ in Γ, counted with their algebraic multiplicity.

Explain the choice of parameters!

Some ideas in the proofs

The random potential is constructed as a finite linear combination of eigenfunctions of an auxiliary elliptic operator with many parameters .....
As in other works, we identify the eigenvalues of the operator with the zeros of a holomorphic function with exponential growth and show that with probability close to 1 this function is exponentially large at finitely many points distributed nicely along the boundary of Γ, then apply a result about the number of zeros of such functions.

First we construct a symbol $\widetilde{p}$, equal to p outside a compact set such that $\widetilde{p} - z \neq 0$ for $z \in \mathrm{neigh}\,(\Gamma)$, and put on the operator level: $\widetilde{P} = P + (\widetilde{p} - p)$. Then $\widetilde{P} - z$ has a bounded (pseudodifferential) inverse for every z in some simply connected neighborhood of Γ. The eigenvalues of P coincide with the zeros of the holomorphic function,

$$z \mapsto \det(\widetilde{P} - z)^{-1}(P - z) = \det(1 - (\widetilde{P} - z)^{-1}(\widetilde{P} - P)).$$

If $P_W = P + W$, put $\widetilde{P}_W := \widetilde{P} + W$ which has no spectrum near Γ. The eigenvalues of P_W in that region are the zeros of

$$z \mapsto \det(\widetilde{P}_{W,z}),$$

where

$$\widetilde{P}_{W,z} = (\widetilde{P}_W - z)^{-1}(P_W - z) = 1 - (\widetilde{P}_W - z)^{-1}(\widetilde{P} - P).$$

The general strategy is the following:

- Step 1. Show that with probability close to 1, we have for all z in a neighborhood of $\partial\Gamma$ with $p_z = (\widetilde{p} - z)^{-1}(p - z)$:
$$\ln|\det P_{W,z}| \leq \frac{1}{(2\pi h)^n}\left(\int \ln|p_z(\rho)|d\rho + o(1)\right). \qquad (11)$$

- Step 2. Show that for each z in a neighborhood of $\partial\Gamma$ we have with probability close to one that
$$\ln|\det P_{W,z}| \geq \frac{1}{(2\pi h)^n}\left(\int \ln|p_z(\rho)|d\rho + o(1)\right). \qquad (12)$$

- Step 3. Apply results ([Ha05, Ha06b, HaSj08, Sj09]) about counting zeros of holomorphic functions: Roughly, if $u(z) = u(z;\widetilde{h})$ is holomorphic with respect to z in a neighborhood of $\overline{\Gamma}$, $|u(z)| \leq \exp(\phi(z)/\widetilde{h})$ near $\partial\Gamma$ and we have a reverse estimate $|u(z_j)| \geq \exp((\phi(z_j) - \text{"small"})/\widetilde{h})$ for a finite set of points, distributed "densely" along the boundary, then the number of zeros of u in Γ is equal to $(2\pi\widetilde{h})^{-1}(\iint_\Gamma \Delta\phi(z)d\Re zd\Im z + \text{"small"})$. This is applied with $\widetilde{h} = (2\pi h)^n$, $\phi(z) = \int \ln|p_z(\rho)|d\rho$.

2. Resonances in semi-classical potential scattering

Consider

$$P = -h^2\Delta + V(x),\ \ V \in L^\infty_{\mathrm{comp}}(\mathbf{R}^n;\mathbf{R})$$

This is a self-adjoint operator with domain $H^2(\mathbf{R}^n)$ and the scattering poles can be defined as the poles of the meromorphic extension of the resolvent $H^0_{\mathrm{comp}}(\mathbf{R}^n) \to H^2_{\mathrm{loc}}(\mathbf{R}^n)$ from $\mathbf{C} \setminus [0,+\infty[$ to the double/logarithmic covering space of $\mathbf{C} \setminus \{0\}$ when n is odd/even.

In one dimension and for $h = 1$ M. Zworski [Zw87] showed that if $[a, b]$ is the convex hull of the support of V, then the number $N(r)$ of resonances in the disc $D(0, r^2)$ satisfies

$$N(r) = \frac{2(b-a)}{\pi} r + o(r), \ \ r \to \infty \tag{13}$$

which is 2 times the asymptotic number of eigenvalues $\leq r^2$ of the Dirichlet realization of $-\Delta + V$ on $[a, b]$. He also showed that most of these concentrate to narrow sectors around the real axis. This extended an earlier result of Regge. Subsequently R. Froese [Fr97] got similar results for potentials that do not necessarily have compact support but are very small near infinity. See also the recent works [DaPu10, DaExLi10, ExLi11] about Weyl and non-Weyl asymptotics for graphs.

In higher odd dimensions, M. Zworski [Zw89] considered the case of radial potentials of the form $V(x) = f(|x|)$ with support in $\overline{B(0,a)}$ where $f \in C^2([0,a])$, $a > 0$, $f(a) \neq 0$ and obtained a Weyl type asympotics (still with $h = 1$)

$$N(r) = K_n a^n r^n + o(r^n), \;\; r \to +\infty, \tag{14}$$

where $K_n > 0$.

P. Stefanov [St06]: Explicit formula for the constant $K_n a^n$ and link to the distribution of eigenvalues for the Dirichlet problem in the ball and for the resonances for the exterior obstacle problem. Also a very precise upper bound for non-radial potentials with support in the ball.

T. Christiansen [Chr10]: Extension and more precise results for a class of non-necessarily radial potentials supported in the ball. Also upper and lower bounds in sectors for “averaged” counting functions for generic potentials with support in the ball.

The main result

Let $\mathcal{O} \Subset \mathbf{R}^n$ be open strictly convex with smooth boundary. Let $N = \min(]\frac{n-1}{2}, +\infty[\cap \mathbf{Z})$, $\widetilde{s} > \max(\frac{n}{2} + 3, 2N + \frac{n}{2})$. The unperturbed operator is:

$$P_0 = -h^2 \Delta + V_0 : \; L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n), \tag{15}$$

$V_0 \in H^{\widetilde{s}}(\overline{\mathcal{O}})$, i.e. $V_0 \in H^{\widetilde{s}}(\mathbf{R}^n)$ and $\mathrm{supp}\,(V_0) \subset \overline{\mathcal{O}}$.
Let P^0_{in} denote the Dirichet realization of P_0 in $\mathcal{O}$ and let $N_0(I)$ denote the number of eigenvalues of P^0_{in} in the interval I.

T. Hargé and G. Lebeau [HaLe94] showed that the exterior Dirichlet problem for $-h^2\Delta$ on $\mathbf{R}^n \setminus \mathcal{O}$ has no resonances in the set

$$\Im z \geq -2(h\Re z)^{\frac{2}{3}}\kappa\zeta_1, \ \frac{1}{2} \leq \Re z \leq 2, \tag{16}$$

if h is small enough and

$$0 < \kappa < 2^{-\frac{1}{3}}\zeta_1 \cos\frac{\pi}{6} \min_{S\partial\mathcal{O}} Q^{\frac{2}{3}}, \tag{17}$$

Q is the second fundamental form on $\partial\mathcal{O}$ and $\zeta_1 > 0$ is the smallest zero of $\mathrm{Ai}(-t)$.

Theorem ([Sj11])

Let $s > \frac{n}{2}$, $\beta > 0$. Then there exists a probability measure μ on $H^s(\overline{\mathcal{O}})$ with support in the ball $\{W \in H^s(\overline{\mathcal{O}});\ \|W\|_{H^s} \leq h^\beta\}$ such that the following holds:
Let $0 < c < 2(1/2)^{2/3}\kappa\zeta_1$, and let $\delta_0 > 0$ be arbitrarily small but fixed. There exist constants C, $N_0 > 0$ such that if $\frac{1}{2} \leq a < b \leq 2$, $\widetilde{\epsilon} \geq Ch(\ln 1/h)^2$ and $V_0 \in H^{\widetilde{s}}(\overline{\mathcal{O}})$, then for $P = -h^2\Delta + V_0 + W$, $W \in H^s(\overline{\mathcal{O}})$, we have with probability (with respect to the random term W)

$$\geq 1 - \mathcal{O}(1)h^{-N_0}e^{-\frac{\widetilde{\epsilon}}{Ch(\ln 1/h)^2}}, \tag{18}$$

that for the set $\sigma(P)$ of resonances of P,

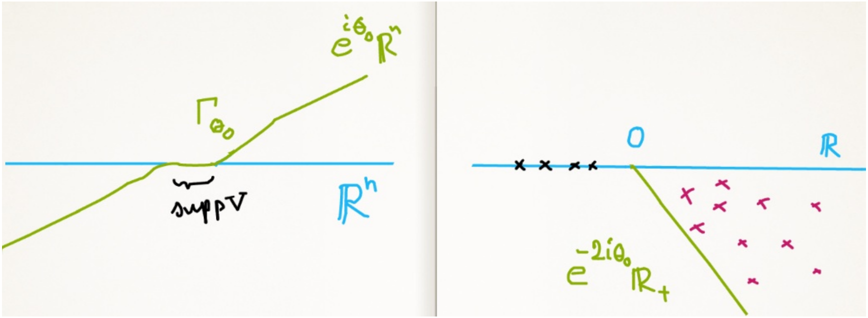
$$\begin{aligned} &|\#(\sigma(P) \cap ([a,b] + ih^{\frac{2}{3}}c[-1,0])) - N_0([a,b])| \\ &\leq \mathcal{O}(1)\Big(\sum_{w=a,b} N_0([w - h^{-\delta_0+2/3}, w + h^{-\delta_0+2/3}])\Big) + h^{-\frac{2}{3}-n}\widetilde{\epsilon}). \end{aligned} \tag{19}$$

Perspectives

1) The restriction to $\Im z \geq -c_0 h^{2/3}$ is probably only due to the proof. Could be replaced by $\Im z \geq -Ch^{2/3}$ or even $\Im z \geq -o(1)$.
2) As in [BoSj10] it should be possible to get a result about the *almost sure* Weyl asymptotics for the *large* resonances of $-\Delta + V_\omega$ in a region $0 < -\Im\lambda \leq C(\Re\lambda)^{2/3}$, $1 \leq \Re\lambda \leq r$, $r \to \infty$.
3) The proof seems to give more general results for operators of black box type, with the black box situated inside $\mathcal{O}$ and with the random perturbation supported near $\partial\mathcal{O}$.
4) One would like to have a more general theorem for $-h^2\Delta + V$ where the convex hull of $\mathrm{supp}\, V$ plays the role of $\mathcal{O}$.
5) There are some interesting upper bounds on the density of resonances close to the real axis related to the Minkowski dimension of the trapped set for the classical flow. To get matching lower bounds, the addition of a small random perturbation is a possibility.

Some elements of the proof

Method of complex scaling (Aguilar-Combes, Balslev-Combes, Simon ...): Close to $\mathbf{R}_+$ the resonances coincide with the eigenvalues of a complex distorsion $P_\Gamma = P_{|_\Gamma}$ as illustrated on the next drawing:



Γ is a maximally totally real submanifold of $\mathbf{C}^n$ equal to $e^{i\theta_0}\mathbf{R}^n$ near infinity. The red crosses illustrate the resonances.

Explanation: P_Γ is an elliptic operator on Γ and the symbol near ∞ is equal to $e^{-i2\theta_0}\xi^2$ so $P_\Gamma - z$ is elliptic near ∞ provided that $z \notin e^{-2i\theta_0}[0, +\infty[$.

The perturbed operator is:

$$P = P_\delta = P_0 + \underbrace{\delta \Theta(x) q_\omega(x)}_{W(x)}, \tag{20}$$

where $\Theta(x) \in C^\infty(\overline{\mathcal{O}})$ is identified with its zero extension and for some $v_0 \in]\frac{n-1}{2}, +\infty[\cap \mathbf{N}$:

$$0 < \Theta(x) \asymp \mathrm{dist}\,(x, \partial\mathcal{O})^{v_0}, \text{ near } \partial\mathcal{O}. \tag{21}$$

q_ω is a random linear combination of eigenfunctions of an auxiliary elliptic operator.

We use a contour $\Gamma \subset \mathbf{C}^n$ which coincides with $\mathbf{R}^n$ along $\mathcal{O}$ and with $e^{i\theta_0}\mathbf{R}^n$ near ∞, $0 < \theta_0 < \pi/2$. Let $P = P_\Gamma$ be the corresponding dilation of $-h^2\Delta + V$, $V = V_0 + \delta\Theta(x)q_\omega(x)$. Then $P = P_\Gamma$ has discrete spectrum in the angle $-2\theta_0 < \arg z \leq 0$ and the eigenvalues there coincide with the resonances.
Let P_{ext} be the Dirichlet realization of P on $\Gamma \setminus \mathcal{O}$, so that the spectrum of P_{ext} in the above angle coincides with the set of resonances for the exterior Dirichlet problem for $-h^2\Delta$ (recalling that $\mathrm{supp}\, V \subset \overline{\mathcal{O}}$).

Restricting z to the domain

$$\frac{1}{2} < \Re z < 2,\ \Im z > -c_0 h^{\frac{2}{3}} \tag{22}$$

and extending suitably the notion of the determinant we get

$$\det \mathcal{P}_{\mathrm{out}}(z) = \det \mathcal{P}_{\mathrm{in}}(z) \det(\mathcal{N}_{\mathrm{in}}(z) - \mathcal{N}_{\mathrm{ext}}(z)). \tag{23}$$

Here:

- $\mathcal{P}_{\mathrm{out}}(z)$ corresponds to our operator on $\mathcal{O}$ with outgoing boundary condition. The zeros of the determinant are the resonances that we are after.
- $\mathcal{P}_{\mathrm{in}}(z)$ corresponds to the Dirichlet problem on $\mathcal{O}$
- $\mathcal{N}_{\mathrm{out}}$ and $\mathcal{N}_{\mathrm{in}}$ are the Dirichlet to Neumann maps for the exterior and interior problems respectively. They are pseudodifferential operators on $\partial\mathcal{O}$.

A rather substantial part of the paper is devoted to the study of $\mathcal{N}_{\rm in}$, $\mathcal{N}_{\rm ext}$, in the regions $|\Im z| \geq h^{2/3}/\widetilde{C}$ and $\Im z \geq -c_0 h^{2/3}$ respectively and we get somewhat roughly,

$$\ln|\det(\mathcal{N}_{\rm in} - \mathcal{N}_{\rm ext})| \leq \mathcal{O}(h^{1-n}). \tag{24}$$

for

$$\Re z \in]\frac{1}{2}, 2[,\ |\Im z| \asymp h^{2/3},\ \Im z \geq -h^{\frac{2}{3}} c_0. \tag{25}$$

The exponent in (24) reflects the fact that we have made a reduction to the $n-1$ dimensional manifold $\partial\mathcal{O}$.

(23) and (24) can be used to prove the upper bound

$$\ln|\det \mathcal{P}_{\rm out}(z)| \le \Phi_{\rm in}(z) + \mathcal{O}(h^{1-n}) \tag{26}$$

in the rectangle $]\frac{1}{2}, 2[+ih^{2/3}]-c_0, c_0[$, where $\Phi_{\rm in}(z)$ coincides with $\ln|\det \mathcal{P}_{\rm in}(z)|$ for $|\Im z| \ge h^{2/3}/\widetilde{C}$ and is extended suitably to the whole rectangle.

In the spirit of [Sj08a, Sj08b]) one can show that for every z with $h^{2/3}/\widetilde{C} \le |\Im z| \le c_0 h^{2/3}$, $1/2 < \Re z < 2$, we also have a lower bound on $\ln|\det(\mathcal{N}_{\rm in} - \mathcal{N}_{\rm ext})|$ almost as sharp as the upper bound (24) with probability very close to 1.

With these upper and lower bounds at our disposal, the main result follows by applying Theorem 1.2 of [Sj09] to the holomorphic function $\det \mathcal{P}_{\rm out}(z)$.

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