Classical Coulomb gases, (Ginzburg-Landau), and the Renormalized Energy

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Spectral days, Munich, 2012

The statistical mechanics of Coulomb gases

Probability law (=Gibbs measure)

$$d\mathbb{P}_n^\beta(x_1, \cdots, x_n) = \frac{1}{Z_n^\beta} e^{-\frac{\beta}{2} w_n(x_1, \cdots, x_n)} dx_1 \cdots dx_n$$

where Z_n^β is the associated partition function, and

$$w_n(x_1, \cdots, x_n) = -\sum_{i \neq j} \log |x_i - x_j| + n \sum_{i=1}^n V(x_i).$$

and $x_i \in \mathbb{R}^d$. V smooth enough and grows faster than $\log^2 |x|$ at infinity.

Important examples

- For general β and V, these ensembles are called Coulomb gases, or sometimes β-ensembles. book by Forrester
- Minimizers of w_n = "weighted Fekete points" (important in interpolation)
- Analogous problems better studied in dimension 3 (Lieb-Narnhofer, Lieb-Oxford...)
- Statistical mechanics (Alastuey-Jancovici, Jancovici-Leibowitz-Manificat, Sari-Merlini, Frölich-Spencer)
- For $d = 1$, $\beta = 2$, $V(x) = x^2/2 \rightsquigarrow$ **GUE** (= law of eigenvalues of Hermitian matrices with complex Gaussian i.i.d. entries).
- For $d = 1$, $\beta = 1$, $V(x) = x^2/2 \rightsquigarrow$ **GOE** (real symmetric matrices with Gaussian i.i.d. entries).
- For $d = 2$, $\beta = 2$ and $V(x) = |x|^2 \rightsquigarrow$ **Ginibre ensemble** (matrices with complex Gaussian i.i.d. entries).

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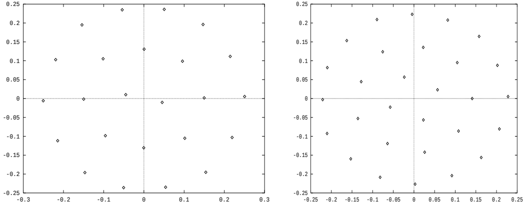
Numerical minimization



Numerical minimization of w_n by Gueron-Shafrir, $n = 24, 29$

Equilibrium measure

Define

$$I(\mu) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} -\log|x-y| \, d\mu(x) \, d\mu(y) + \int_{\mathbb{R}^2} V(x) \, d\mu(x).$$

I has a unique minimizer among probability measures, called the *equilibrium measure*, denoted μ_0.

Denote $E = Supp(\mu_0)$ (assumed to be compact with C^1 boundary).

- For GUE: $\mu_0 = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{\{|x|<2\}}$ (Wigner semicircle law, Deift, Ben Arous-Guionnet).
- For Ginibre: $\mu_0 = \frac{1}{\sqrt{\pi}}\mathbf{1}_{B_1}$ (circle law, Edelman, Girko, Mehta).

Large Deviations

$$\frac{w_n}{n^2}\ \Gamma - \text{converges to } I$$

i.e.

$$\lim_{n\to\infty} \frac{\min w_n}{n^2} = I(\mu_0) \qquad \lim_{n\to\infty} \frac{\sum_{i=1}^n \delta_{x_i}}{n} = \mu_0 \quad \text{for a minimizer}$$

Theorem (Ben Arous-Guionnet $d = 1$, Ben Arous-Zeitouni $d = 2$, Hiai-Petz)

$\mathbb{P}_n^\beta$ satisfies a large deviations principle with good rate function $I(\cdot)$ and speed n^{-2}: for all $A \subset \{$probability measures$\}$,

$$-\inf_{\mu\in A^\circ} \widetilde{I}(\mu) \le \liminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n^\beta(A)$$

$$\le \limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n^\beta(A) \le -\inf_{\mu\in \bar{A}} \widetilde{I}(\mu),$$

where $\widetilde{I} = I - \min I$.

First objective

Understand minimizers of w_n. We know the global distribution of the points and $\min w_n \sim n^2 I(\mu_0)$.
Can we say more about the local distribution of points and the next order terms in $\min w_n$?? For that we want to blow up the points at the scale $\sqrt{n}$ to see them at finite distances from each other.

Splitting of w_n

The idea is to understand the next order behavior by splitting w_n, writing $\nu_n := \sum_{i=1}^n \delta_{x_i}$ as $n\mu_0 + (\nu_n - n\mu_0)$. We find

$$w_n(x_1, \cdots, x_n) = n^2 I(\mu_0) + \frac{1}{\pi} W(\nabla H, \mathbf{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^n \zeta(x_i)$$

where H is the solution to

$$H = -2\pi \Delta^{-1} \left(\sum_{i=1}^n \delta_{x_i} - n\mu_0 \right)$$

and

$$\begin{cases} \zeta = cst + \frac{1}{2} V - \int \log|x-y| \, d\mu_0(y) & \\ \zeta = 0 & \text{in } E \\ \zeta > 0 & \text{in } \mathbb{R}^2 \backslash E \end{cases}$$

and for every function χ,

$$W(\nabla H, \chi) := \lim_{\eta \to 0} \int_{\mathbb{R}^2 \backslash \cup_{i=1}^n B(x_i, \eta)} \chi |\nabla H|^2 + \pi \log \eta \sum_i \chi(x_i).$$

In rescaled coordinates $x' = \sqrt{n}(x - x_0)$ this becomes

$$w_n(x_1, \cdots, x_n) = n^2 I(\mu_0) - \frac{n}{2}\log n + \frac{1}{\pi} W(\nabla H', \mathbf{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^{n} \zeta(x_i)$$

where H' is the solution to

$$H'(x') = -2\pi\Delta^{-1}\left(\sum_{i=1}^{n} \delta_{x'_i} - \mu_0(x_0 + \frac{x'}{\sqrt{n}})\right)$$

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- remains to understand this $W(\nabla H', \mathbf{1}_{\mathbb{R}^2})$, "renormalized" Coulomb interaction between the points in a neutralizing background, of slowly varying density $\sim \mu_0$
- difficulties in letting $n \to \infty$, in particular no local "charge neutrality"
- need to define a total Coulomb interaction for such a system with infinite number of points

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Complete definition of W

Let $m > 0$ given. Let Λ be a discrete set in $\mathbb{R}^2$, and $j(= \nabla H)$ a vector field such that

$$\operatorname{div} j = 2\pi(\nu - m) \quad \text{and} \quad \operatorname{curl} j = 0, \quad \text{where} \quad \nu = \sum_{p \in \Lambda} \delta_p.$$

We say such a j belongs to the class $\mathcal{A}_m$.

Definition

For any smooth positive χ, let

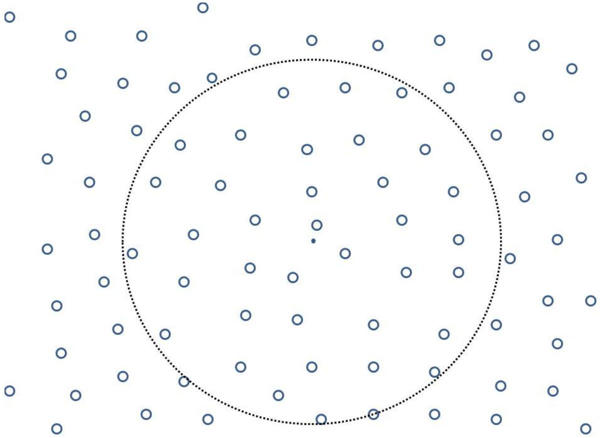
$$W(j,\chi) = \lim_{\eta \to 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p \in \Lambda} B(p,\eta)} \chi |j|^2 + \pi \log \eta \sum_{p \in \Lambda} \chi(p) \right).$$

We define the **renormalized energy** W *by*

$$W(j) = \limsup_{R \to \infty} \frac{W(j, \chi_{B_R})}{|B_R|},$$

where χ_{B_R} is any cutoff function supported in B_R with $\chi_{B_R} = 1$ in B_{R-1} and $|\nabla \chi_{B_R}| \le C$.

Computing W



The case of the torus

Assume Λ is $\mathbb{T}$-periodic. Then W can be written as a function of Λ. Identify Λ with $\{a_1, \ldots, a_n\} \subset \mathbb{T}$. Let $H_{\{a_i\}}$ be a solution of

$$-\Delta H_{\{a_i\}} = 2\pi \left(\sum_{i=1}^{n} \delta_{a_i} - \frac{n}{|\mathbb{T}|} \right) \quad \text{on } \mathbb{T}.$$

Let $j_{\{a_i\}} = \nabla H_{\{a_i\}}$, identified with a periodic vector field on $\mathbb{R}^2$.

Lemma

Take the normalization $n = |\mathbb{T}|$. Let G be the Green function for $\mathbb{T}$:

$$-\Delta G(x) = 2\pi \left(\delta_0 - \frac{1}{|\mathbb{T}|} \right) \qquad \textit{in } \mathbb{T},$$

normalized to have mean zero. Then

$$W(j_{\{a_i\}}) = \frac{\pi}{|\mathbb{T}|} \sum_{i \neq j} G(a_i - a_j) + \pi \lim_{x \to 0} \left(G(x) + \log|x| \right).$$

Moreover, $j_{\{a_i\}}$ is the minimizer of $W(j)$ over all $\mathbb{T}$-periodic j satisfying $\operatorname{div} j = 2\pi(\nu - 1)$ and $\operatorname{curl} j = 0$.

Further expression of W in the square torus case

Let $\mathbb{T}_N = \mathbb{R}^2/(N\mathbb{Z})^2$. By a Fourier series expansion, the Green function of $\mathbb{T}_N$ is expressed in terms of Eisenstein series. We obtain:

Proposition

Let $a_1, \cdots, a_n$ be $n = N^2$ points on $\mathbb{T}_N$, we have

$$W(j_{\{a_i\}}) = \frac{1}{2N^2} \sum_{j \neq k} E(a_j - a_k) + \pi \log \frac{N}{2\pi} - 2\pi \log \eta(i).$$

Here $E(x) = E_{\Re(x/N), \Im(x/N)}(i)$ where $E_{u,v}(\tau)$ is the Eisenstein series defined for $\tau \in \mathbb{C}$ and $u, v \in \mathbb{R}$ by

$$E_{u,v}(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} e^{2i\pi(mu+nv)} \frac{\Im(\tau)}{|m\tau + n|^2}.$$

Finally, η denotes the Dedekind η function, which is given by

$$\eta(\tau) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k) \qquad \text{where } q = e^{2i\pi\tau}.$$

Minimization of W

- W is unchanged by a compact perturbation of the point configuration.
- Minimizers of W exist (requires work)
- Scaling: call $\mathcal{A}_m$ the vector fields corresponding to density m, that is, $\mathrm{div}\, j = 2\pi(\nu - m)$ and $\mathrm{curl}\, j = 0$. Then if j belongs to $\mathcal{A}_m$, then $j' = \frac{1}{\sqrt{m}} j(\cdot/\sqrt{m})$ belongs to $\mathcal{A}_1$ and

$$W(j) = m\left(W(j') - \frac{\pi}{2}\log m\right)$$

  so we can reduce to $\mathcal{A}_1$.
- Proposition: $\min_{\mathcal{A}_1} W$ is the limit as $N \to \infty$ of the min over $\mathbb{T}_N$-periodic configurations.

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Minimization among lattices

We can look for minimizers of W among perfect lattice configurations, i.e., $\Lambda = \mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$, with unit volume.

Theorem (Sandier-S. '10)

The minimum of $\Lambda \mapsto W(\Lambda)$ over perfect lattice configurations is achieved uniquely, modulo rotations, by the triangular lattice.

- in that setting, explicit formula in terms of Eisenstein series
- by transformations using modular functions or by direct computations, minimizing W becomes equivalent to minimizing the Epstein zeta function $\zeta(s) = \sum_{p \in \Lambda} \frac{1}{|p|^s}$, $s > 2$, over lattices
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Conjecture

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Γ-convergence of w_n / estimate of ground state energy

Theorem (Sandier-S '12)

Fix $1 < p < 2$ *and let* $X = E \times L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$.

A. Lower bound. *Let* $\nu_n = \sum_{i=1}^n \delta_{x_i}$ *be a sequence such that*

$$w_n(x_1, \cdots, x_n) - n^2 I(\mu_0) + \frac{n}{2} \log n \leq Cn.$$

Let P_n *be the probability on* X *which is the push-forward of* $\frac{1}{|E|} dx_{|E}$ *by*

$$x \mapsto \left(x, j_n(\sqrt{n}x + \cdot)\right), \qquad j_n := \nabla H_n'.$$

1. *Up to a subsequence,* P_n *converges to a probability* P *on* X.
2. *The first marginal of* P *is* $\frac{1}{|E|} dx_{|E}$. P *is invariant by* $(x, j) \mapsto (x, j(\lambda + \cdot))$, *for any* $\lambda \in \mathbb{R}^2$.
3. *For* P *a.e.* (x, j) *we have* $j \in \mathcal{A}_{\mu_0(x)}$.
4. $$\liminf_{n \to \infty} \frac{1}{n} \left(w_n(x_1, \ldots, x_n) - n^2 I(\mu_0) + \frac{n}{2} \log n \right) \geq \frac{|E|}{\pi} \int W(j)\, dP(x, j).$$

Theorem (continued)

B. Upper bound construction. *Conversely, assume P is an invariant probability measure on X whose first marginal is $\frac{1}{|E|}dx_{|E}$ and such that for P-a.e. (x,j) we have $j \in \mathcal{A}_{\mu_0(x)}$. Then there exists a sequence $\{\nu_n = \sum_{i=1}^n \delta_{x_i}\}_n$ of empirical measures on E and a sequence $\{j_n\}_n$ in $L^p_{loc}(\mathbb{R}^2,\mathbb{R}^2)$ such that $\mathrm{div}\, j_n = 2\pi(\nu_n' - \mu_0')$ and such that the image P_n of $\frac{1}{|E|}dx_{|E}$ by $x \mapsto (x, j_n(\sqrt{n}x + \cdot))$ converges to P. Moreover*

$$\limsup_{n\to\infty} \frac{1}{n}\left(w_n(x_1,\ldots,x_n) - n^2 I(\mu_0) + \frac{n}{2}\log n\right) \leq \frac{|E|}{\pi}\int W(j)\, dP(x,j).$$

C. Consequences for minimizers. *Assume $(x_1,\ldots,x_n)$ minimize w_n and let $\nu_n = \sum_{i=1}^n \delta_{x_i}$. Then P_n converges to P, and for P a.e. (x,j), j minimizes W over $\mathcal{A}_{\mu_0(x)}$. Moreover,*

$$\limsup_{n\to\infty} \frac{1}{n}\left(w_n(x_1,\ldots,x_n) - n^2 I(\mu_0) + \frac{n}{2}\log n\right) = \min \frac{|E|}{\pi}\int W(j)\, dP(x,j),$$

and $\lim_{n\to\infty} \sum_{i=1}^n \mathit{dist}^{\,2}(x_i, E) = 0.$

Method of the proof

- Γ-convergence: prove general (ansatz-free) lower bounds and upper bounds which match
- Introduce a new general method for lower bound on two-scale energies (after splitting + blow-up, the domain becomes of infinite size, so it is difficult to localize energy lower bounds). A probability measure approach allows to do this via the use of the ergodic theorem (idea of Varadhan)
- That method applies well to positive (or bounded below) energy densities, but those associated to $W(\nabla H, \chi)$ are not!
- Start by modifying the energy density to make it bounded below: method of mass transport, using sharp energy lower bounds by "ball construction" methods (à la Jerrard / Sandier)

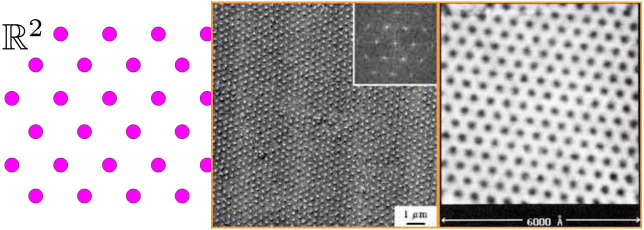
Original motivation: the Ginzburg-Landau model

W is derived as a limit problem for the minimization of the $2D$ Ginzburg-Landau functional of superconductivity:

$$G_\varepsilon(\psi, A) = \frac{1}{2}\int_\Omega |(\nabla - iA)\psi|^2 + |\mathrm{curl}\, A - h_{\mathrm{ex}}|^2 + \frac{(1-|\psi|^2)^2}{2\varepsilon^2}.$$

- $\psi : \Omega \to \mathbb{C}$ "order parameter"
- $|\psi|^2$= density of superconducting Cooper pairs, $|\psi| \sim 1$ superconducting phase, $|\psi| \sim 0$ normal phase, $\psi = 0$ *vortices*
  $\psi = \rho e^{i\varphi}$
  $$\frac{1}{2\pi}\int_{\partial B(x_0, r)} \frac{\partial \varphi}{\partial \tau} = d \in \mathbb{Z}$$
  degree of the vortex
- $A : \Omega \to \mathbb{R}^2$ vector potential $\nabla_A = \nabla - iA$
- $h = \mathrm{curl}\, A$ induced magnetic field
- $h_{\mathrm{ex}} > 0$ intensity of applied field
- $\varepsilon = \frac{1}{\kappa}$ "Ginzburg-Landau parameter": material constant
- limit $\varepsilon \to 0$ extreme type-II or strongly repulsive

For $H_{c_1} < h_{\mathrm{ex}} \ll \frac{1}{\varepsilon^2}$, minimizers (ψ, A) of G_ε have vortices ($=$ zeros of the complex-valued function ψ) which are densely-packed in the domain.



Abrikosov lattice

From Ginzburg-Landau to W

Minimizing the Ginzburg-Landau energy is roughly equivalent to minimizing

$$\frac{1}{2}\int_\Omega |\nabla h_\varepsilon|^2 + |h_\varepsilon - h_{\mathrm{ex}}|^2$$

with

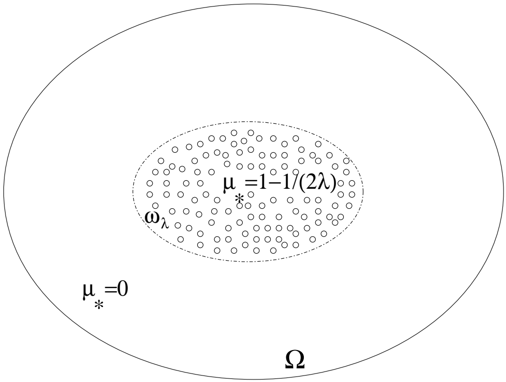
$$\begin{cases} -\Delta h_\varepsilon + h_\varepsilon = \mu_\varepsilon \simeq 2\pi \sum_i d_i \delta_{a_i}^{(\varepsilon)} & \text{in } \Omega \\ h_\varepsilon = h_{\mathrm{ex}} & \text{on } \partial\Omega. \end{cases}$$

$\delta_{a_i}^{(\varepsilon)}$ Dirac mass regularized at the scale ε

Mean field description for $h_{\rm ex} > H_{c_1}$ (Sandier-S)

$$h_{\rm ex} = \lambda |\log \varepsilon|, \quad \lambda > \lambda_\Omega \qquad H_{c_1} \sim \lambda_\Omega |\log \varepsilon|$$

$\dfrac{\mu_\varepsilon}{h_{\rm ex}} \rightharpoonup \mu_*$ solution to an obstacle problem



Theorem (Sandier-S. '10)

Consider minimizers $(u_\varepsilon, A_\varepsilon)$ *of the Ginzburg-Landau. After blow-up around a randomly chosen point in* ω_λ*, their "currents"* $\nabla h_\varepsilon (= \nabla \mathrm{curl}\, A_\varepsilon)$ *converge as* $\varepsilon \to 0$ *to currents in the plane which, almost surely, minimize* W*. Moreover,*

$$\min G_\varepsilon = h_{\mathrm{ex}}^2 E_\lambda(\mu_*) + (1 - 1/(2\lambda)) h_{\mathrm{ex}} |\omega_\lambda| (\min W + \gamma) + o(h_{\mathrm{ex}})$$

Method: same as for Coulomb gases, but complicated by the presence of vortices of arbitrary signs and degrees

Back to Coulomb gases: Next-order expansion of the partition function

Theorem (Sandier-S. '12)

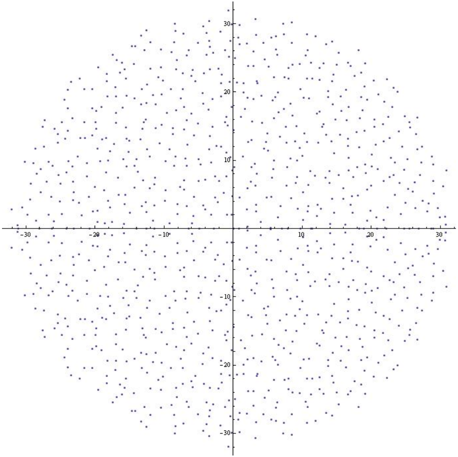
$$n\beta f_1(\beta) \le \log Z_n^\beta - \Big(-\beta n^2 I(\mu_0) + \frac{\beta n}{2} \log n \Big) \le n\beta f_2(\beta),$$

where $f_1(\beta)$ and $f_2(\beta)$ are independent of n, bounded, and

$$\lim_{\beta\to\infty} f_1(\beta) = \lim_{\beta\to\infty} f_2(\beta) = \alpha_0,$$

where

$$\alpha_0 = \frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int \mu_0 \log \mu_0 \, dx.$$



Eigenvalues of 1000-by-1000 matrix with i.i.d Gaussian entries

(Stolen from Benedek Valkó's webpage)

Large deviations type result

Theorem (Sandier-S.)

Let $A_n \subset (\mathbb{R}^2)^n$. Then

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n^\beta(A_n) \le -\beta\Big(\frac{|E|}{\pi} \inf_{P\in A} \int W(j)\,dP(x,j) - \alpha_0 - \frac{C}{\beta}\Big),$$

and A is the set of probability measures which are limits of blow-ups at rate $n^{1/2}$ around a point x of the current j associated to $\nu = \sum_{i=1}^n \delta_{x_i}$ with $(x_i) \in A_n$.

Corollary: crystallisation as $\beta \to \infty$: $\rightsquigarrow$ after blowing up around a point x in the support of μ_0, at the scale of $(n\mu_0(x))^{1/2}$, we see (almost surely) a configuration which minimizes W.

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and A *is the set of probability measures which are limits of blow-ups at rate* $n^{1/2}$ *around a point* x *of the current* j *associated to* $\nu = \sum_{i=1}^n \delta_{x_i}$ *with* $(x_i) \in A_n$.

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Definition of $\mathcal{W}$ for arbitrary point processes (with Borodin)

By analogy with the $\mathbb{T}_N$-periodic case, we define for any point process the random variable

$$\mathcal{W}_N = -\frac{1}{N} \sum_{i \neq j, a_i, a_j \in [0,N]} \log \left| 2 \sin \frac{\pi(a_i - a_j)}{N} \right| + \log N \qquad \text{in dimension 1,}$$

and

$$\mathcal{W}_N = \frac{1}{2\pi N^2} \sum_{i \neq j, a_i, a_j \in [0,N]^2} E(a_i - a_j) + \log \frac{N}{2\pi \eta(i)^2} \qquad \text{in dimension 2.}$$

For stationary processes, we give conditions for $\mathbb{E}\mathcal{W}_N$ to have a limit as $N \to \infty$ as well as for $\mathrm{Var}\mathcal{W}_N$.

Characterization of the expectation of W

Theorem (Borodin-S.)

Let a random point process in $\mathbb{R}^d$ ($d = 1$ or 2) have two-point correlation function $\rho_2(x, y) = 1 - T_2(x - y)$. If $\int T_2 = 1$ and T_2 satisfies some decay conditions, then

$$\lim_{N\to\infty} \mathbb{E}\mathcal{W}_N = \int_{\mathbb{R}^d} \log|2\pi v|\, T_2(v)\, dv.$$

Moreover, under additional decay conditions, $\lim_{N\to\infty} \mathrm{Var}\mathcal{W}_N = 0$.

Examples

- Poisson process in dimensions $d = 1, 2$: $\lim_{N\to\infty} \mathbb{E}\mathcal{W}_N = +\infty$.
- perfect lattice $\mathbb{Z}$ in dimension $d = 1$: $\lim_{N\to\infty} \mathbb{E}\mathcal{W}_N = \mathbb{E}\mathcal{W}_N = 0$.
- sine-beta process in dimension $d = 1$:

$$\begin{cases} \lim\limits_{N\to\infty} \mathbb{E}\mathcal{W}_N = 2 - \gamma - \log 2 & \beta = 1, \\ \lim\limits_{N\to\infty} \mathbb{E}\mathcal{W}_N = 1 - \gamma & \beta = 2, \\ \lim\limits_{N\to\infty} \mathbb{E}\mathcal{W}_N = \frac{3}{2} - \gamma - \log 2 & \beta = 4, \end{cases}$$

Directly related to the "thermodynamic energy per particle" for the log gas found in Dyson '62, Dyson-Mehta '63, $\mathcal{W}$ provides the rigorous quantity.

- The determinantal process ($d = 2$) with kernel $e^{-\frac{\pi}{2}|x-y|^2}$:

$$\lim_{N\to\infty} \mathbb{E}\mathcal{W}_N = \frac{1}{2}(\gamma - \log \pi).$$

- Zeros of Gaussian Analytic Functions:

$$\lim_{N\to\infty} \mathbb{E}\mathcal{W}_N = -\frac{1}{2}(1 + \log \pi)$$

Extensions

- extension of the definition of W to 1D and analogous results (with E. Sandier).
  In 1D, $\min W$ is achieved by the perfect lattice $\mathbb{Z}$, and the crystallisation result is complete.
- usual Fekete points on a compact set (with A. Contreras and E. Sandier)
- quantum Coulomb gases in 2D (with M. Lewin and P. T. Nam).