Local behavior of solutions of Schrödinger equations and bounds on the density of states for Schrödinger operators

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Ergodic Schrödinger operators

Let $H_\omega = -\Delta + V_\omega$ be an ergodic Schrödinger operator on either $\mathrm{L}^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$, and consider its integrated density of states $N(E)$.

What can we say about the continuity of $N(E)$?

- Craig and Simon (1983) proved log-Hölder continuity for discrete ergodic Schrödinger operators and one-dimensional ergodic Schrödinger operators :

$$N(E+\varepsilon) - N(E) \le \frac{C}{\log \frac{1}{\varepsilon}} \quad \text{for} \quad \varepsilon \le \tfrac{1}{2}.$$

- Delyon and Souillard (1984) gave a simple proof of continuity of $N(E)$ for discrete ergodic Schrödinger operators.

In this generality not more appears to have been known.

Random Schrödinger operators

For Anderson models with regularity assumptions we have Wegner estimates. The optimal result is due to Combes, Hislop and Klopp (2007), who proved for both discrete and continuous Anderson models that

$$N(E+\varepsilon) - N(E) \le C\, S_\mu(\varepsilon) \quad \text{for} \quad \varepsilon \le \tfrac{1}{2},$$

where $S_\mu(s) := \sup_{t\in\mathbb{R}} \mu([t, t+s])$ is the concentration function of the single-site probability distribution μ.

Note that

$$\mu \quad \text{has no atoms} \quad \Longleftrightarrow \quad \lim_{s\downarrow 0} S_\mu(s) = 0 \quad \Longrightarrow \quad \text{continuity of} \quad N(E)$$

The integrated density of states $N(E)$ is always continuous when the single-site probability distribution μ has no atoms.

Multi-dimensional continuous Anderson models

What could we say about the continuity of the density states for multi-dimensional continuous Anderson models with arbitrary single-site probability distribution (e.g., Bernoulli)?

- Germinet and Klein proved log-Hölder continuity of $N(E)$ in the region of localization (more precisely, in the region of applicability of the multiscale analysis):

$$N(E+\varepsilon)-N(E) \leq \frac{C}{\left(\log \frac{1}{\varepsilon}\right)^{\frac{3}{8}d-}} \quad \text{for} \quad \varepsilon \leq \tfrac{1}{2}.$$

  This result holds for Poisson Hamiltonians.
- In general the continuity of the integrated density of states has been an open question.
- Continuity of the integrated density of states for (continuous) ergodic Schrödinger operators is # 14 in Simon's Problems for Schrödinger operators.

Schrödinger operators

We consider the Schrödinger operator

$$H = -\Delta + V \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d) \quad \Big(\text{or} \quad \ell^2(\mathbb{Z}^d)\Big),$$

where Δ is the Laplacian operator and V is a bounded potential.

We let

$$\Lambda_L(x) := x + \left]-\tfrac{L}{2}, \tfrac{L}{2}\right[^d$$

denote the (open) box of side L centered at $x \in \mathbb{R}^d$. By a box Λ_L we will mean a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$.

Given a finite open box $\Lambda \subset \mathbb{R}^d$ we let H_Λ and Δ_Λ be the restriction of H and Δ to $\mathrm{L}^2(\Lambda)$ with Dirichlet boundary condition. (Our results will be independent of the boundary condition.)

Density of states measures and outer-measures

We define finite volume (normalized) density of states measures on Borel subsets B of R^d by

$$\eta_\Lambda(B) = \eta_{\Lambda,\infty}(B) := \tfrac{1}{|\Lambda|} \operatorname{tr}\{\chi_B(H)\chi_\Lambda\},$$
$$\eta_{\Lambda,D}(B) := \tfrac{1}{|\Lambda|} \operatorname{tr}\{\chi_B(H_\Lambda)\}$$

Note that for $\sharp = \infty, D$ and $B \subset]-\infty, E]$ we have

$$\eta_{\Lambda,\sharp}(B) \le C_{d,V_\infty,E} < \infty.$$

We define outer-measures on Borel subsets B of R^d for $\sharp = \infty, D$ by

$$\eta^*_{L,\sharp}(B) := \sup_{x\in\mathbb{R}^d} \eta_{\Lambda_L(x),\sharp}(B),$$
$$\eta^*_\sharp(B) := \limsup_{L\to\infty} \eta^*_{L,\sharp}(B).$$

Density of states outer-measures

For any two values of $\sharp$ we have (e.g., Doi, Iwatsuka and Mine (2001))

$$\eta^*_{\sharp_1}([E_1,E_2]) \le \eta^*_{\sharp_2}([E_1-\delta,E_2+\delta]) \quad \text{for all} \quad \delta>0.$$

Thus, if for some value of $\sharp$ we have

$$\lim_{\varepsilon\to 0} \eta^*_{\sharp}([E-\varepsilon,E+\varepsilon]) = 0 \quad \text{for all} \quad E\in\mathbb{R},$$

we conclude that for all $E_1,E_2\in\mathbb{R}$, $E_1\le E_2$, we have

$$\eta^*([E_1,E_2]) := \eta^*_\infty([E_1,E_2]) = \eta^*_D([E_1,E_2]).$$

Remark

For ergodic Schrödinger operators H_ω, if η is the density of states measures, i.e., $\eta(]E_2,E_2] = N(E_2)-N(E_1)$, we always have

$$\eta\,(]E_2,E_2]) \le \eta^*_\omega([E_1,E_2]) \quad \textit{with probability one} \quad .$$

Main theorems

Theorem (Discrete Schrödinger operators)

Let H *be a Schrödinger operator on* $\ell^2(\mathbb{Z}^d)$*. Then for all* $E \in \mathbb{R}$ *and* $\varepsilon \le \frac{1}{2}$ *we have*

$$\eta^*([E, E+\varepsilon]) \le \frac{C_{d,\|V\|_\infty}}{\log \frac{1}{\varepsilon}}.$$

Theorem (Continuous Schrödinger operators)

Let H *be a Schrödinger operator on* $\mathrm{L}^2(\mathbb{R}^d)$*, where* $d = 1, 2, 3$*. Then, given* $E_0 \in \mathbb{R}$*, for all* $E \le E_0$ *and* $\varepsilon \le \frac{1}{2}$ *we have*

$$\eta^*([E, E+\varepsilon]) \le \frac{C_{d,\|V\|_\infty,E_0}}{\left(\log \frac{1}{\varepsilon}\right)^{\kappa_d}}, \quad \text{where} \quad \kappa_1 = 1,\ \kappa_2 = \tfrac{1}{4},\ \kappa_3 = \tfrac{1}{8}.$$

Comments

- The theorems are proved in finite volume with Dirichlet boundary condition.
- The discrete case and the one-dimensional continuous case are proved in the same way. We select approximate eigenfunctions for which we have global upper bounds, and pick one for which we have a local lower bound.
- The two and three-dimensional continuous case require a different approach. We select approximate eigenfunctions for which we have local upper bounds, and pick one for which we have a global lower bound.
- Our proof will give log-Hölder continuity of the density states in any dimension in the continuous case if we can improve on the exponent in the quantitative unique continuation principle.

Discrete case and the one-dimensional continuous case

Theorem

Let H *be a discrete Schrödinger operator. Let* $E \in \mathbb{R}$ *and* $0 < \varepsilon \leq \frac{1}{2}$ *. Then for all boxes* $\Lambda = \Lambda_L$ *with* $L \geq L_{d,V_\infty} \log \frac{1}{\varepsilon}$ *we have*

$$\eta_\Lambda([E, E+\varepsilon]) \leq \frac{C_{d,V_\infty}}{\log \frac{1}{\varepsilon}}.$$

Theorem

Let H *be a one-dimensional Schrödinger operator. Given* $E_0 \in \mathbb{R}$*, there exists* L_{V_∞,E_0} *such that for all* $0 < \varepsilon \leq \frac{1}{2}$*, open intervals* $\Lambda = \Lambda_L$ *with* $L \geq L_{V_\infty,E_0} \log \frac{1}{\varepsilon}$*, and energies* $E \leq E_0$*, we have*

$$\eta_\Lambda([E, E+\varepsilon]) \leq \frac{C_{V_\infty,E_0}}{\log \frac{1}{\varepsilon}}.$$

Proof for the one-dimensional continuous case

Let $\Lambda = \Lambda_L =]a_0, a_0 + L[$, $E \in \mathbb{R}$, $\varepsilon \in]0, \frac{1}{2}]$, and set $P = \chi_{[E,E+\varepsilon]}(H_\Lambda)$. We have $\operatorname{Ran} P \subset \mathscr{D}(\Delta_\Lambda) \subset C^1(\Lambda)$ and

$$\|(H_\Lambda - E)\,\psi\| \le \varepsilon\,\|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P.$$

Given $0 < R < L$, set $a_j = a_0 + jR$ for $j = 1, 2, \ldots, \frac{L}{R}$. We introduce the vector space

$$F_R := \left\{\psi \in \operatorname{Ran} P;\ \ \psi(a_j) = \psi'(a_j) = 0 \text{ for } j = 1, 2, \ldots, \tfrac{L}{R} - 1\right\}.$$

If $\psi \in F_R$, It follows from Gronwall's inequality that for all $x \in]a_j - R, a_j + R[$ we have, with $K = 1 + \|V - E\|_\infty$,

$$|\psi(x)| \le \mathrm{e}^{K|x-a_j|}\left|\int_{a_j}^{x} \mathrm{e}^{-K|y-a_j|}\,|(H_\Lambda - E)\,\psi(y)|\,\mathrm{d}y\right| \le (2K)^{-\frac{1}{2}}\,\mathrm{e}^{KR}\varepsilon\,\|\psi\|.$$

Proof for the one-dimensional continuous case -continued

Since Λ is the union of these intervals, we conclude that

$$\|\psi\|_\infty \leq (2K)^{-\frac{1}{2}} \mathrm{e}^{KR} \varepsilon \|\psi\| \quad \text{for all} \quad \psi \in F_R.$$

We now assume that

$$\eta_\Lambda([E, E+\varepsilon]) = \tfrac{1}{L} \operatorname{tr} P \geq \rho > 0.$$

If $R \in [\frac{4}{\rho}, L]$, it follows that $\dim F_R \geq \rho L - 2\frac{L}{R} \geq \frac{1}{2}\rho L$.

A useful lemma

Lemma

Let $\mathscr{V}$ be a finite dimensional linear subspace of $\mathrm{L}^{\infty}(\Omega, \mathbb{P})$, *where* $(\Omega, \mathbb{P})$ *is a probability space.*
Then there exists $\psi \in \mathscr{V}$ *with* $\|\psi\|_2 = 1$ *such that*

$$\|\psi\|_{\infty} \geq \sqrt{\dim \mathscr{V}}.$$

This lemma follows from the theory of absolutely summing operators. The lemma can also be proved by a direct argument.

Proof for the one-dimensional continuous case–end

Applying the Lemma, we obtain $\psi \in F_R$, $\psi \neq 0$, such that

$$\|\psi\|_\infty \geq \sqrt{\frac{\dim F_R}{L}}\,\|\psi\| \geq \sqrt{\tfrac{1}{2}\rho}\,\|\psi\|\,.$$

Taking $R = \frac{4}{\rho}$ and $L > R$, it follows that

$$\sqrt{\tfrac{1}{2}\rho} \leq (2K)^{-\frac{1}{2}}\,\mathrm{e}^{KR}\varepsilon = (2K)^{-\frac{1}{2}}\,\mathrm{e}^{\frac{4K}{\rho}}\,\varepsilon.$$

Thus we get

$$\rho \leq \frac{C_K}{\log\frac{1}{\varepsilon}}\,.$$

and hence

$$\eta^*([E, E+\varepsilon]) \leq \frac{C_K}{\log\frac{1}{\varepsilon}}\,.$$

The discrete case has a similar proof.

Key ingredients for the multidimensional case

We select approximate eigenfunctions for which we have local upper bounds, and pick one for which we have a global lower bound.

1. Local behavior with bounds for approximate solutions of Schrödinger equations
   We extend the local behavior results of Hartman and Wintner (1953) and Bers (1955) for solutions of Schrödinger equations to obtain local behavior with bounds for approximate solutions.
2. The quantitative unique continuation principle
   We use a version of Bourgain and Kenig's quantitative unique continuation principle, rewritten to explicit the dependence on the relevant parameters.

Local behavior of approximate solutions for $d \geq 2$

Theorem

Let $\Omega = B(x_0, r_0)$ *for some* $x_0 \in \mathbb{R}^d$ *and* $r_0 > 0$*, where* $d = 2, 3, \ldots$. *Given* $W \in \mathrm{L}^\infty(\Omega)$ *real-valued, let* $\mathscr{F}$ *be a linear subspace of* $\mathscr{H}^2(\Omega)$ *with the following property:*

$$\|(-\Delta + W)\,\psi\|_{\mathrm{L}^\infty(\Omega)} \leq C_{\mathscr{F}}\,\|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{for all} \quad \psi \in \mathscr{F}.$$

Then there exist constants $\gamma_d > 0$ *and* $0 < r_1 = r_1\,(d, \|W\|_\infty) < r_0$*, with the property that for all* $N \in \mathbb{N}$ *there is a linear subspace* $\mathscr{F}_N$ *of* $\mathscr{F}$*,with*

$$\dim \mathscr{F}_N \geq \dim \mathscr{F} - \gamma_d\,N^{d-1},$$

such that for all $\psi \in \mathscr{F}_N$ *we have*

$$|\psi(x)| \leq \left(C_{d,\|W\|_\infty}^{N^2}\,|x - x_0|^{N+1} + C_{\mathscr{F}} \right) \|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{for} \quad x \in B(x_0, r_1).$$

Local behavior of solutions for $d \geq 2$

Let $d \geq 2$, $x_0 \in \mathbb{R}^d$, $r_0 > 0$, $W \in \mathrm{L}^\infty(\Omega)$ real-valued, and consider the equation

$$-\Delta\phi + W\phi = 0 \quad \text{on} \quad \Omega = B(x_0, 2r_0).$$

Let $\mathscr{E}_0(\Omega)$ be the space of solutions $\phi \in \mathscr{H}^2(\Omega)$. Define linear subspaces

$$\mathscr{E}_N(\Omega) = \left\{ \phi \in \mathscr{E}_0(\Omega);\ \limsup_{x \to x_0} \frac{|\phi(x)|}{|x - x_0|^N} < \infty \right\} \quad \text{for} \quad N \in \mathbb{N}.$$

$\mathscr{E}_1(\Omega) = \{\phi \in \mathscr{E}_0(\Omega);\ \phi(x_0) = 0\}$, $\mathscr{E}_N(\Omega) \supset \mathscr{E}_{N+1}(\Omega)$.
(Note $\cap_{N=0}^\infty \mathscr{E}_N(\Omega) = \{0\}$ by the unique continuation principle.)

Let $\mathscr{H}_m^{(d)}$ denote the vector space of homogenous harmonic polynomials on $\mathbb{R}^d$ of degree m, and recall that

$$\sum_{m=0}^{N} \dim \mathscr{H}_m^{(d)} \leq \gamma_d N^{d-1} \quad \text{for all} \quad N \in \mathbb{N}$$

Lemma (Local behavior of solutions in $d \geq 2$)

Let $\Omega = B(x_0, 2r_0)$ for some $x_0 \in \mathbb{R}^d$ and $r_0 > 0$.
For all $N \in \mathbb{N}_0$ there is a linear map $Y_N^{(\Omega)} \colon \mathscr{E}_N(\Omega) \to \mathscr{H}_N^{(d)}$ such that for all $\phi \in \mathscr{E}_N(\Omega)$ we have

$$\left| \phi(x) - \left(Y_N^{(\Omega)} \phi \right)(x - x_0) \right| \leq C_{d,r_0,W_\infty}^{N^2} |x - x_0|^{N+1} \|\phi\|_{\mathrm{L}^2(\Omega)}$$

for all $x \in \overline{B}(x_0, \frac{r_0}{2})$, where $W_\infty = \|W\|_{\mathrm{L}^\infty(\Omega)}$.
As a consequence, for all $N \in \mathbb{N}_0$ we have

$$\mathscr{E}_{N+1}(\Omega) = \ker Y_N^{(\Omega)} \quad \text{and} \quad \dim \mathscr{E}_{N+1}(\Omega) \geq \dim \mathscr{E}_N(\Omega) - \dim \mathscr{H}_N^{(d)}.$$

In particular, if $\mathscr{J}$ is a vector subspace of $\mathscr{E}_0(\Omega)$ we have

$$\dim \mathscr{J} \cap \mathscr{E}_{N+1}(\Omega) \geq \dim \mathscr{J} - \gamma_d N^{d-1} \quad \text{for all} \quad N \in \mathbb{N}.$$

Lemma (Local behavior of approximate solutions in $d \geq 2$)

Let $\Omega = B(x_0, r_1)$, $x_0 \in \mathbb{R}^d$, $r_1 > 0$.
Let $\mathscr{F} \subset \mathscr{H}^2(\Omega)$ linear subspace with

$$\|(-\Delta + W)\psi\|_{\mathrm{L}^\infty(\Omega)} \leq \varepsilon \|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{for all} \quad \psi \in \mathscr{F}.$$

Then there is $0 < r_2 = r_2(d, W_\infty) < r_1$, with the property that for all $r \in]0, r_2]$ there exists a linear map $Z_r \colon \mathscr{F} \to \mathscr{E}_0(B(x_0, r))$ such that

$$\|\psi - Z_r\psi\|_{\mathrm{L}^\infty(B(x_0,r))} \leq C_{d,r}\varepsilon \|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{where} \quad \lim_{r \to 0} C_{d,r} = 0.$$

As a consequence, for all $N \in \mathbb{N}$ there is a vector subspace $\mathscr{F}_N$ of $\mathscr{F}$, with

$$\dim \mathscr{F}_N \geq \dim \mathscr{F} - \gamma_d N^{d-1},$$

such that for all $\psi \in \mathscr{F}_N$ and $x \in \overline{B}(x_0, \frac{r_2}{4})$ we have

$$|\psi(x)| \leq \left(C_{d,W_\infty,r_1}^{N^2} |x - x_0|^{N+1} + \varepsilon \right) \|\psi\|_{\mathrm{L}^2(\Omega)}.$$

Theorem (Quantitative unique continuation principle)

Let $\Omega \subset \mathbb{R}^d$ open. Let $\psi \in \mathrm{H}^2(\Omega)$ and let $\zeta \in \mathrm{L}^2(\Omega)$ be defined by

$$-\Delta\psi + V\psi = \zeta \quad \text{a.e. on} \quad \Omega,$$

where V is a bounded real measurable function on Ω, $\|V\|_\infty \le K < \infty$.
Let $\Theta \subset \Omega$ be a bounded measurable set where $\left\|\psi\chi_\Theta\right\|_2 > 0$.

$$\text{Set} \quad Q(x,\Theta) := \sup_{y\in\Theta} |y-x| \quad \text{for} \quad x \in \Omega.$$

$$\text{Let} \quad x_0 \in \Omega\setminus\overline{\Theta} \quad \text{satisfy} \quad Q = Q(x_0,\Theta) \ge 1 \quad \text{and} \quad B(x_0, 6Q+2) \subset \Omega.$$

Then, given

$$0 < \delta \le \min\left\{2\,\mathrm{dist}\,(x_0,\Theta)\,,\tfrac{1}{300}\right\},$$

we have

$$\left(\frac{\delta}{Q}\right)^{m_d\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\|\psi\chi_\Omega\|_2}{\|\psi\chi_\Theta\|_2}\right)} \|\psi\chi_\Theta\|_2^2 \le \left\|\psi\chi_{B(x_0,\delta)}\right\|_2^2 + \|\zeta\chi_\Omega\|_2^2 .$$

Bounds for two and three dimensional Schrödinger operators

Theorem

Let H *be a Schrödinger operator in* $\mathrm{L}^2(\mathbb{R}^d)$*, where* $d = 2, 3$*.*
Given $E_0 \in \mathbb{R}$*, there exists* L_{d,V_∞,E_0} *such that for all energies* $E \leq E_0$ *and* $0 < \varepsilon \leq \frac{1}{2}$ *we have*

$$\eta_{\Lambda_L}([E, E+\varepsilon]) \leq \frac{C_{d,V_\infty,E_0}}{\left(\log \frac{1}{\varepsilon}\right)^{\frac{4-d}{8}}}$$

for all open boxes Λ_L *with* $L \geq L_{d,V_\infty,E_0}\left(\log \frac{1}{\varepsilon}\right)^{\frac{3}{8}}$*,*

Proof for the two and three dimensional continuous cases I

Let $\Lambda = \Lambda_L(x_0)$, $E \leq E_0$, $\varepsilon \in]0, \frac{1}{2}]$, and set $P = \chi_{[E,E+\varepsilon]}(H_\Lambda)$.
Recall that $\operatorname{Ran} P \subset \mathscr{D}(\Delta_\Lambda) \subset \mathscr{H}^2(\Lambda)$, and note that we have

$$\|(H_\Lambda - E)\,\psi\| \leq \varepsilon\,\|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P.$$

Recall also that for $\psi \in \operatorname{Ran} P$ we have

$$\|\psi\|_\infty = \left\|\mathrm{e}^{-H_\Lambda}\mathrm{e}^{H_\Lambda}\psi\right\|_\infty \leq \left\|\mathrm{e}^{-H_\Lambda}\right\|_{\mathrm{L}^2\to\mathrm{L}^\infty}\left\|\mathrm{e}^{H_\Lambda}\psi\right\| \leq C_{d,V_\infty}\mathrm{e}^{|E|+1}\,\|\psi\|\,.$$

Since $P\,(H_\Lambda - E)\,\psi = (H_\Lambda - E)\,P\psi$ for $\psi \in \operatorname{Ran} P$, we conclude that

$$\|(H_\Lambda - E)\,\psi\|_\infty \leq \varepsilon\,C_{d,V_\infty,E_0}\,\|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P.$$

Proof for the two and three dimensional continuous cases II

Suppose now that

$$\eta_\Lambda([E, E+\varepsilon]) = \tfrac{1}{L^d} \operatorname{tr} P \geq \rho > 0.$$

We fix $0 < R < L$, to be selected later, and let $\mathscr{G} \subset \Lambda$ be defined by

$$\overline{\Lambda} = \bigcup_{y \in \mathscr{G}} \overline{\Lambda}_R(y) \quad \text{and} \quad \#\mathscr{G} = \left(\tfrac{L}{R}\right)^d.$$

We take

$$N \approx \left(\tfrac{1}{2\gamma_d}\rho\right)^{\frac{1}{d-1}} R^{\frac{d}{d-1}} \in \mathbb{N}.$$

Applying the local behavior theorem at all the sites in $\mathscr{G}$, we conclude that there exists a subspace $\mathscr{F}_R$ of $\operatorname{Ran} P$ and $r_0 = r_0(d, V_\infty, E) > 0$, such that

$$\dim \mathscr{F}_R \geq \rho L^d - \gamma_d N^{d-1} \left(\tfrac{L}{R}\right)^d \geq \tfrac{1}{2}\rho L^d,$$

and for all $\psi \in \mathscr{F}_R$ and $y \in \mathscr{G}$ we have

$$|\psi(y+x)| \leq \left(C_{d,V_\infty,E}^{N^2} |x|^{N+1} + \varepsilon C_{d,V_\infty,E}\right) \|\psi\| \quad \text{if} \quad |x| \leq r_0.$$

Proof for the two and three dimensional continuous cases III

Let Q_R be the orthogonal projection onto $\mathscr{F}_R$, so $Q_R = Q_R P$.
Since $\operatorname{tr} Q_R = \dim \mathscr{F}_R \geq \frac{1}{2}\rho L^d$, there is $\psi \in \mathscr{F}_R$, $\|\psi\| = 1$, and $x_0 \in \Lambda$ such that $\Lambda_1(x_0) \subset \Lambda$ and

$$\left\|\psi\chi_{\Lambda_1(x_0)}\right\|_2 \geq \gamma\rho, \quad \text{where} \quad \gamma = \gamma(d, V_\infty, E_0) > 0.$$

We then pick $y_0 \in \mathscr{G}$ such that

$$\tfrac{R}{2} \leq \operatorname{dist}(y_0, \Lambda_1(x_0)) \leq R\sqrt{d}$$

Taking $0 < \delta < \frac{1}{300}$, it follows from the QUCP that, with $K = \|V\|_\infty + E_0$,

$$\left(\tfrac{\delta}{\sqrt{d}R}\right)^{m_d\left(1+K^{\frac{2}{3}}\right)\left(R^{\frac{4}{3}}+\left|\log\left\|\psi\chi_{\Lambda_1(x_0)}\right\|_2\right|\right)}\left\|\psi\chi_{\Lambda_1(x_0)}\right\|_2^2 \leq \left\|\psi\chi_{B(y_0,\delta)}\right\|_2^2 + \varepsilon^2.$$

Making $\delta \leq r_0$ and using the local behavior estimate, we get

$$\left(\tfrac{\delta}{\sqrt{d}R}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(R^{\frac{4}{3}}+|\log(\gamma\rho)|\right)}(\gamma\rho)^2 \leq C_d\, C_{d,V_\infty,E_0}^{N^2}\, \delta^{2(N+1)+d} + C_{d,V_\infty,E_0}\, \varepsilon^2.$$

Proof for the two and three dimensional continuous cases IV

Using $\rho R^d \geq 2\gamma_d$ (it follows from $N \geq 1$), noting $\delta \leq d^{-\frac{1}{2}} R$, we get

$$\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4}{3}}} \leq C_{d,V_\infty,E_0}^{N^2} \delta^{2N} + C_{d,V_\infty,E_0}\, \varepsilon^2,$$

with $M = M_{d,V_\infty,E_0} > 0$.

We now choose δ by

$$C_{d,V_\infty,E_0}^{N} \delta^2 = \tfrac{\delta}{R}, \quad \text{i.e.,} \quad \delta = \left(C_{d,V_\infty,E_0}^{N} R\right)^{-1},$$

obtaining

$$\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4}{3}}} \leq \left(\tfrac{\delta}{R}\right)^{N} + C_{d,V_\infty,E_0}\, \varepsilon^2.$$

Proof for the two and three dimensional continuous cases V

We now take $d = 2, 3$ and take R large enough so that

$$\left(\frac{\delta}{R}\right)^{N} \le \frac{1}{2}\left(\frac{\delta}{R}\right)^{MR^{\frac{4}{3}}},$$

which can always be done since for $d = 2, 3$ we have $\frac{4}{3} < \frac{d}{d-1}$, so

$$MR^{\frac{4}{3}} < N \approx \left(\frac{1}{2\gamma_d}\rho\right)^{\frac{1}{d-1}} R^{\frac{d}{d-1}}, \quad \text{i.e.,} \quad \rho > C_{d,V_\infty,E} R^{\frac{d-4}{3}}.$$

(Here we need $d < 4$, that is, $d = 2$ or $d = 3$.)
It follows that

$$\frac{1}{2}\left(\frac{\delta}{R}\right)^{MR^{\frac{4}{3}}} \le C_{d,V_\infty,E_0}\,\varepsilon^2.$$

We conclude that

$$\left(C^N_{d,V_\infty,E} R^2\right)^{-MR^{\frac{4}{3}}} \le 2C_{d,V_\infty,E_0}\,\varepsilon^2.$$

Proof for the two and three dimensional continuous cases VI

We now choose R (with an appropriate constant $c_{d,V_\infty,E}$) by

$$\rho = c_{d,V_\infty,E_0} R^{\frac{d-4}{3}},$$

We get

$$\mathrm{e}^{-M' R^{\frac{8}{3}}} = \mathrm{e}^{-M' R^{\frac{d-4}{3(d-1)} + \frac{d}{d-1} + \frac{4}{3}}} \le C_{d,V_\infty,E_0}\, \varepsilon^2 \quad \text{where} \quad M' = M'_{d,V_\infty,E_0}.$$

Thus

$$\log \tfrac{1}{\varepsilon} \le C_{d,V_\infty,E_0} R^{\frac{8}{3}} = \frac{C'_{d,V_\infty,E_0}}{\rho^{\frac{8}{4-d}}},$$

and hence

$$\rho \le \frac{C_{d,V_\infty,E_0}}{\left(\log \frac{1}{\varepsilon}\right)^{\frac{4-d}{8}}}.$$

What about $d \geq 3$?

- The restriction to $d = 1, 2, 3$ is due to the present form of the quantitative unique continuation principle, where there is a term $Q^{\frac{4}{3}}$ in the exponent.
- If instead we had Q^{β} we would be able to prove log-Hölder continuity of the integrated density of states for dimensions $d < \frac{\beta}{\beta-1}$. Since $\beta = \frac{4}{3}$, we get $d < 4$.
- It is reasonable to expect that something like the the quantitative unique continuation principle holds with $\beta = 1+$ (there are no counterexamples for real potentials). In this case we would obtain log-Hölder continuity of the integrated density of states for all d, with

$$\kappa_d = \frac{\beta - d(\beta - 1)}{2\beta} = \frac{1}{2}- \quad \text{for} \quad d \geq 2.$$