Correlated Markov Quantum Walks *

Alain JOYE

INSTITUT FOURIER

* Joint work with Eman HAMZA (Cairo University)

Annales Henri Poincaré, to appear

Quantum Walk

Unitary evolution: Particle with spin ("coin") on d-dim lattice

Setup: $\mathcal{H} = \mathbb{C}^{2d} \otimes l^2(\mathbb{Z}^d)$

$\{|\tau\rangle\}_{\tau \in I_\pm}$, $I_\pm \equiv \{\pm 1, \pm 2, \ldots, \pm d\}$ for $\mathbb{C}^{2d}$,

$\{|k\rangle\}_{k \in \mathbb{Z}^d}$ for $l^2(\mathbb{Z}^d)$

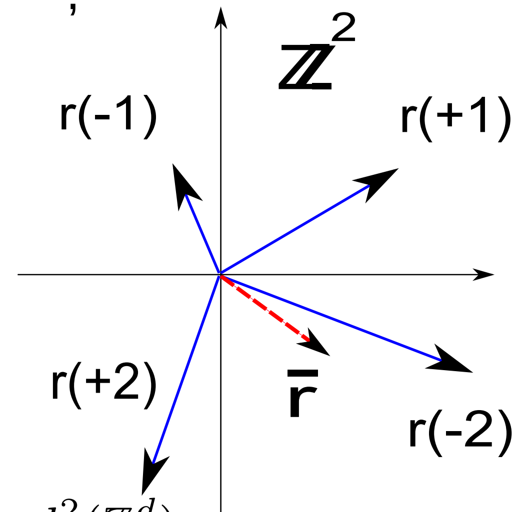
Quantum Walk

Unitary evolution: Particle with spin ("coin") on d-dim lattice

Setup: $\mathcal{H} = \mathbb{C}^{2d} \otimes l^2(\mathbb{Z}^d)$

$\{|\tau\rangle\}_{\tau \in I_\pm}$, $I_\pm \equiv \{\pm 1, \pm 2, \dots, \pm d\}$ for $\mathbb{C}^{2d}$,

$\{|k\rangle\}_{k \in \mathbb{Z}^d}$ for $l^2(\mathbb{Z}^d)$



Ingredients:

Jump funct: $r : I_\pm \to \mathbb{Z}^d$ s.t. $\tau \mapsto r(\tau)$

Coin dep. shift: Let P_τ the proj. "on" $|\tau\rangle \in \mathbb{C}^{2d}$

$S := \sum_{x \in \mathbb{Z}^d} \sum_{\tau \in I_\pm} P_\tau \otimes |x + r(\tau)\rangle\langle x|$ on $\mathbb{C}^{2d} \otimes l^2(\mathbb{Z}^d)$

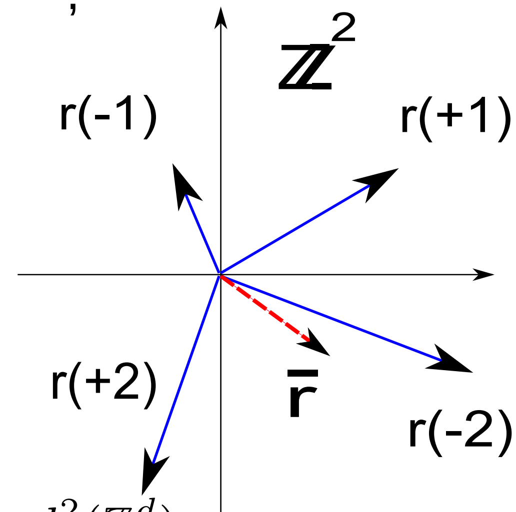
Quantum Walk

Unitary evolution: Particle with spin ("coin") on d-dim lattice

Setup: $\mathcal{H} = \mathbb{C}^{2d} \otimes l^2(\mathbb{Z}^d)$

$\{|\tau\rangle\}_{\tau \in I_\pm}$, $I_\pm \equiv \{\pm 1, \pm 2, \dots, \pm d\}$ for $\mathbb{C}^{2d}$,

$\{|k\rangle\}_{k \in \mathbb{Z}^d}$ for $l^2(\mathbb{Z}^d)$



Ingredients:

Jump funct: $r : I_\pm \to \mathbb{Z}^d$ s.t. $\tau \mapsto r(\tau)$

Coin dep. shift: Let P_τ the proj. "on" $|\tau\rangle \in \mathbb{C}^{2d}$

$S := \sum_{x \in \mathbb{Z}^d} \sum_{\tau \in I_\pm} P_\tau \otimes |x + r(\tau)\rangle\langle x|$ on $\mathbb{C}^{2d} \otimes l^2(\mathbb{Z}^d)$

Coin evol.: For a config. $C = \{C(x)\}_{x \in \mathbb{Z}^d}$ of unitary op. on $\mathbb{C}^{2d}$

Time one dynamics of the QW:

$$U := S(C \otimes \mathbb{I}) = \sum_{x \in \mathbb{Z}^d} \sum_{\tau \in I_\pm} (P_\tau C(x)) \otimes |x + r(\tau)\rangle\langle x|$$

$U = \sum_{x\in\mathbb{Z}^d} \sum_{\tau\in I_\pm} (P_\tau C(x)) \otimes |x + r(\tau)\rangle\langle x|$

Dynamics: U^n , $n \in \mathbb{Z}$. Behaviour of U^n , $n \to \infty$?

Position op. $X = \mathbb{I} \otimes x$ on $\mathcal{H}$, with $x|k\rangle = k|k\rangle$ on $l^2(\mathbb{Z}^d)$

Init. cond. $\psi_0 = \varphi_0 \otimes |0\rangle \in \mathbb{C}^{2d} \otimes l^2(\mathbb{Z}^d)$, $\|\varphi_0\|_{\mathbb{C}^{2d}} = 1$

Periodicity $\exists\, \Gamma \subset \mathbb{Z}^d$ s.t. $C(x+\gamma) = C(x)$, $\forall\, \gamma \in \Gamma \subset \mathbb{Z}^d$.

Ballistic behaviour: "Generically"

$$\boxed{\langle X^2(n)\rangle_{\psi_0} := \langle U^n\psi_0|\, X^2\, U^n\psi_0\rangle_{\mathcal{H}} \simeq n^2 \quad n \to \infty.}$$

$$U = \sum_{x\in\mathbb{Z}^d} \sum_{\tau\in I_\pm} (P_\tau C(x)) \otimes |x + r(\tau)\rangle\langle x|$$

Dynamics: U^n, $n \in \mathbb{Z}$. Behaviour of U^n, $n \to \infty$?

Position op. $X = \mathbb{I} \otimes x$ on $\mathcal{H}$, with $x|k\rangle = k|k\rangle$ on $l^2(\mathbb{Z}^d)$

Init. cond. $\psi_0 = \varphi_0 \otimes |0\rangle \in \mathbb{C}^{2d} \otimes l^2(\mathbb{Z}^d)$, $\|\varphi_0\|_{\mathbb{C}^{2d}} = 1$

Periodicity $\exists\, \Gamma \subset \mathbb{Z}^d$ s.t. $C(x+\gamma) = C(x)$, $\forall\, \gamma \in \Gamma \subset \mathbb{Z}^d$.

Ballistic behaviour: "Generically"

$$\boxed{\langle X^2(n)\rangle_{\psi_0} := \langle U^n\psi_0|\, X^2\, U^n\psi_0\rangle_{\mathcal{H}} \simeq n^2 \quad n \to \infty.}$$

- Implemented experimentally '09 Meschede et al, '10 Zähringer et al
  Atoms in a tunable optical lattice
- Effective quantum dynamics '88 Chalker Coddington, '96 Meyer
  for "networks systems"
- Used in Quantum Information Theory '03 Shenvi et al, '08 Santha

Time Dependent Random Quantum Walk

Discrete time $j \in \mathbb{N}^*$

Random Coin op's

Sequence of config. $\{C_j^\omega\}^{j\in\mathbb{N}} = \{C_j^\omega(x)\}_{x\in\mathbb{Z}^d}^{j\in\mathbb{N}}$

Deterministic Jump Function

$r : I_\pm \to \mathbb{Z}^d$ s.t. $\tau \mapsto r(\tau)$

Time Dependent Random Quantum Walk

Discrete time $\quad j \in \mathbb{N}^*$

Random Coin op's

Sequence of config. $\{C_j^\omega\}^{j\in\mathbb{N}} = \{C_j^\omega(x)\}_{x\in\mathbb{Z}^d}^{j\in\mathbb{N}}$

Deterministic Jump Function

$r : I_\pm \to \mathbb{Z}^d$ s.t. $\tau \mapsto r(\tau)$

Random evol.

$$U_\omega(n,0) \quad = \quad S(C_n^\omega \otimes \mathbb{I}) \cdots S(C_2^\omega \otimes \mathbb{I})\, S(C_1^\omega \otimes \mathbb{I})$$

- Mimicks Anderson model with random time-dependent potential
- $S \simeq \Delta$ and $C_j^\omega(x) \simeq V_j^\omega(x)$
- Simpler dynamics

'74 Ovchinnikov, Erikhman '85 Pillet, '09-10-11 Schenker et al

Correlated Markov Distribution à la Hamza, Kang, Schenker '10

Prob. space $\Omega = \{C_1, C_2, \ldots, C_F\}, \quad C_j \in U(2d)$

Markov chain on Ω $\omega = \{\omega(1), \omega(2), \omega(3), \cdots\} \in \Omega^{\mathbb{N}^*}$

Transition kernel $\mathbb{P}(\xi, \eta) = \mathbb{P}(\omega(n) = \eta \,|\omega(n-1) = \xi)$

Initial stationary distrib. $p(\eta) = \mathbb{P}(\omega(0) = \eta)$

Correlated Markov Distribution   à la Hamza, Kang, Schenker '10

Prob. space $\Omega = \{C_1, C_2, \dots, C_F\}, \quad C_j \in U(2d)$

Markov chain on Ω $\omega = \{\omega(1), \omega(2), \omega(3), \cdots\} \in \Omega^{\mathbb{N}^*}$

Transition kernel $\mathbb{P}(\xi, \eta) = \mathbb{P}(\omega(n) = \eta \,|\omega(n-1) = \xi)$

Initial stationary distrib. $p(\eta) = \mathbb{P}(\omega(0) = \eta)$

Measure Preserving Representation of $\mathbb{Z}^d$ $\sigma : \mathbb{Z}^d \to \text{Bij}(\Omega)$

$x \mapsto \sigma_x$

i.e. $\sigma_{x+y} = \sigma_x \cdot \sigma_y \,, \quad \sigma_0 = \mathbb{I}$

and $\mathbb{P}(\sigma_x \xi, \sigma_x \eta) = \mathbb{P}(\xi, \eta), \quad p(\sigma_x \eta) = p(\eta)$

Correlated Markov Distribution à la Hamza, Kang, Schenker '10

Prob. space $\Omega = \{C_1, C_2, \ldots, C_F\}, \;\; C_j \in U(2d)$

Markov chain on Ω $\omega = \{\omega(1), \omega(2), \omega(3), \cdots\} \in \Omega^{\mathbb{N}^*}$

Transition kernel $\mathbb{P}(\xi, \eta) = \mathbb{P}(\omega(n) = \eta \,|\omega(n-1) = \xi)$

Initial stationary distrib. $p(\eta) = \mathbb{P}(\omega(0) = \eta)$

Measure Preserving Representation of $\mathbb{Z}^d$ $\sigma : \mathbb{Z}^d \to \text{Bij}(\Omega)$

$x \mapsto \sigma_x$

i.e. $\sigma_{x+y} = \sigma_x \cdot \sigma_y \,, \quad \sigma_0 = \mathbb{I}$

and $\mathbb{P}(\sigma_x \xi, \sigma_x \eta) = \mathbb{P}(\xi, \eta), \quad p(\sigma_x \eta) = p(\eta)$

Random coin matrices Given σ , $\forall x \in \mathbb{Z}^d$ and $\forall n \in \mathbb{N}^*$

$$\boxed{C_n^\omega(x) = \sigma_x(\omega(n))}$$

- $C_n^\omega(x)$ is Γ-periodic, $\forall n \in \mathbb{N}^*$, where $\Gamma = \{x \mid \sigma_x = \mathbb{I}\}$
- Distrib. of $C_n^\omega(x)$ is that of $\omega(0)$, $\quad \forall n \in \mathbb{N}^*, x \in \mathbb{Z}^d$

Density Matrices on $l^2(\mathbb{Z}^d; \mathbb{C}^{2d})$

$$\rho = \sum_{(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d} \rho(x,y) \otimes |x\rangle\langle y| \,, \quad \rho(x,y) \in M_{2d}(\mathbb{C})$$

$\rho \simeq \rho(x,y)$ with $\rho \in l^2(\mathbb{Z}^d \times \mathbb{Z}^d; M_{2d}(\mathbb{C}))$

$$\rho_0 = |\varphi_0\rangle\langle\varphi_0| \otimes |0\rangle\langle 0| \simeq \rho_0(x,y) = \delta_0(x) \otimes \delta_0(y) \otimes |\varphi_0\rangle\langle\varphi_0|$$

$\Rightarrow \quad \rho_n^\omega = U_\omega(n,0)\rho_0 U_\omega^*(n,0) \simeq \rho_n^\omega(x,y)$ cpct supp.

Density Matrices on $l^2(\mathbb{Z}^d;\mathbb{C}^{2d})$

$$\rho = \sum_{(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d} \rho(x,y)\otimes|x\rangle\langle y|\,, \quad \rho(x,y)\in M_{2d}(\mathbb{C})$$

$$\boxed{\rho \simeq \rho(x,y) \text{ with } \rho\in l^2(\mathbb{Z}^d\times\mathbb{Z}^d; M_{2d}(\mathbb{C}))}$$

$$\rho_0 = |\varphi_0\rangle\langle\varphi_0|\otimes|0\rangle\langle 0| \simeq \rho_0(x,y) = \delta_0(x)\otimes\delta_0(y)\otimes|\varphi_0\rangle\langle\varphi_0|$$

$$\Rightarrow \quad \rho_n^\omega = U_\omega(n,0)\rho_0 U_\omega^*(n,0) \simeq \rho_n^\omega(x,y) \quad \text{cpct supp.}$$

Lattice observables — If $f:\mathbb{Z}^d\to\mathbb{C}$, let $F := \mathbb{I}\otimes f$ on $D(F)\subset\mathcal{H}$ s.t.

$$F(\varphi_0\otimes|k\rangle) = f(k)(\varphi_0\otimes|k\rangle), \quad \forall\varphi_0\in\mathbb{C}^{2d}.$$

Averaged QM Expectation Value — Let $\psi_0 = \varphi_0\otimes|0\rangle$

$$\mathbb{E}_\omega\langle F\rangle_{\psi_0}^\omega(n) = \mathbb{E}_\omega \sum_{x\in\mathbb{Z}^d} \mathrm{Tr}_{M_{2d}(\mathbb{C})}(\rho_n^\omega(x,x))f(x)$$

Density Matrices on $l^2(\mathbb{Z}^d;\mathbb{C}^{2d})$

$$\rho = \sum_{(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d} \rho(x,y)\otimes|x\rangle\langle y|\,,\quad \rho(x,y)\in M_{2d}(\mathbb{C})$$

$$\boxed{\rho \simeq \rho(x,y) \text{ with } \rho\in l^2(\mathbb{Z}^d\times\mathbb{Z}^d; M_{2d}(\mathbb{C}))}$$

$$\rho_0 = |\varphi_0\rangle\langle\varphi_0|\otimes|0\rangle\langle 0| \simeq \rho_0(x,y) = \delta_0(x)\otimes\delta_0(y)\otimes|\varphi_0\rangle\langle\varphi_0|$$

$$\Rightarrow\quad \rho_n^\omega = U_\omega(n,0)\rho_0 U_\omega^*(n,0) \simeq \rho_n^\omega(x,y) \quad \text{cpct supp.}$$

Lattice observables   If $f:\mathbb{Z}^d\to\mathbb{C}$, let $F := \mathbb{I}\otimes f$ on $D(F)\subset\mathcal{H}$ s.t.

$$F(\varphi_0\otimes|k\rangle) = f(k)(\varphi_0\otimes|k\rangle),\quad \forall\varphi_0\in\mathbb{C}^{2d}.$$

Averaged QM Expectation Value   Let $\psi_0 = \varphi_0\otimes|0\rangle$

$$\mathbb{E}_\omega\langle F\rangle_{\psi_0}^\omega(n) = \mathbb{E}_\omega\sum_{x\in\mathbb{Z}^d}\mathrm{Tr}_{M_{2d}(\mathbb{C})}(\rho_n^\omega(x,x))f(x) = \sum_{x\in\mathbb{Z}^d}\underbrace{\mathbb{E}_\omega\mathrm{Tr}_{M_{2d}(\mathbb{C})}(\rho_n^\omega(x,x))}f(x)$$

$$\equiv \mathbb{E}_{w(n)}f(X_n),\quad \text{with}\quad \boxed{X_n \simeq \text{Prob}(X_n = x) := w_x(n).}$$

Ballistic vs. Diffusive Scaling

Goal Understand $\{w_x(n)\}_{x\in\mathbb{Z}^d}$ as $n \to \infty$

Tool Given $X_n \simeq \{w_x(n)\}_{x\in\mathbb{Z}^d}$, and $y \in \mathbb{T}^d$ let

$$\boxed{\Phi_n(y) = \mathbb{E}_{w(n)}(e^{iyX_n}) = \sum_{x\in\mathbb{Z}^d} w_x(n) e^{iyx}}$$ s.t.

$$-i\partial_{y_j}\Phi_n(y/n^\alpha)\big|_{y=0} = \sum_{x\in\mathbb{Z}^d} w_x(n) x_j/n^\alpha = \mathbb{E}_{w(n)}((X_n)_j/n^\alpha)$$

$\alpha = 1 \Leftrightarrow$ ballistic scaling $\quad \alpha = 1/2 \Leftrightarrow$ diffusive scaling

Rem: $\Phi_n(y)$ is analytic in $\mathbb{C}^d$.

Extended Hilbert space $\mathcal{K} = l^2(\mathbb{Z}^d \times \mathbb{Z}^d \times \Omega; M_{2d}(\mathbb{C}))$ with

$$\langle \psi, \varphi \rangle_{\mathcal{K}} = \sum_{\eta \in \Omega} \sum_{x,y \in \mathbb{Z}^d} p(\eta) \mathrm{Tr}_{M_{2d}\mathbb{C}}(\psi^*(x,y,\eta)\varphi(x,y,\eta))$$

Theorem There exists $M : \mathcal{K} \to \mathcal{K}$ s.t. for any ρ_0

$$\boxed{\mathbb{E}_\omega(\rho_n^\omega(x,y)_{\tau,\tau'}) = \langle \delta_x \otimes \delta_y \otimes |\tau\rangle\langle\tau'| \ , \ M^n \rho_0 \rangle_{\mathcal{K}}}$$

where

$$(M\rho)(x,y,\eta) = \sum_{\substack{\tau,\tau' \in I_\pm \\ \zeta \in \Omega}} \mathbb{Q}(\eta,\zeta) P_\tau(\sigma_{x-r(\tau)}\eta)\rho(x-r(\tau), y-r(\tau'), \zeta)(\sigma_{y-r(\tau')}\eta)^* P_{\tau'}$$

and $\mathbb{Q}(\eta,\zeta) = \frac{p(\zeta)}{p(\eta)} \mathbb{P}(\zeta,\eta)$

Extended Hilbert space $\mathcal{K} = l^2(\mathbb{Z}^d \times \mathbb{Z}^d \times \Omega; M_{2d}(\mathbb{C}))$ with

$$\langle \psi, \varphi \rangle_{\mathcal{K}} = \sum_{\eta \in \Omega} \sum_{x,y \in \mathbb{Z}^d} p(\eta) \mathrm{Tr}_{M_{2d}\mathbb{C}}(\psi^*(x,y,\eta)\varphi(x,y,\eta))$$

Theorem There exists $M : \mathcal{K} \to \mathcal{K}$ s.t. for any ρ_0

$$\boxed{\mathbb{E}_\omega(\rho_n^\omega(x,y)_{\tau,\tau'}) = \langle \delta_x \otimes \delta_y \otimes |\tau\rangle\langle\tau'| \;,\; M^n \rho_0 \rangle_{\mathcal{K}}}$$

where

$$(M\rho)(x,y,\eta) = \sum_{\substack{\tau,\tau' \in I_\pm \\ \zeta \in \Omega}} \mathbb{Q}(\eta,\zeta) P_\tau (\sigma_{x-r(\tau)}\eta) \rho(x - r(\tau), y - r(\tau'), \zeta)(\sigma_{y-r(\tau')}\eta)^* P_{\tau'}$$

and $\mathbb{Q}(\eta,\zeta) = \frac{p(\zeta)}{p(\eta)} \mathbb{P}(\zeta,\eta)$

Symmetries For any $\xi \in \mathbb{Z}^d$, $\gamma \in \Gamma = \{x \mid \sigma_x = \mathbb{I}\}$, M commutes with

$$\begin{aligned} (S_\xi \, \rho)(x,y,\eta) &= \rho(x-\xi, y-\xi, \sigma_\xi \eta) \\ (S_\gamma^{(1)} \rho)(x,y;\eta) &= \rho(x-\gamma, y, \eta) \end{aligned}$$

Spectral Analysis

"Fourier" $\mathcal{F} : l^2(\mathbb{Z}^d \times \mathbb{Z}^d \times \Omega; M_{2d}(\mathbb{C}) \to L^2(B_\Gamma \times \mathbb{T}^d \times \mathbb{T}^d_\Gamma \times \Omega; M_{2d}(\mathbb{C}))$

$$\Psi(x, y, \zeta) \mapsto \widehat{\Psi}(x_0, k, p, \zeta) = \sum_{\substack{\xi \in \mathbb{Z}^d \\ \gamma \in \Gamma}} e^{ip\cdot(x-\eta)-ik\cdot\xi} \Psi(x - \xi - \gamma, -\xi, \sigma_\xi \zeta),$$

with $x_0 \in B_\Gamma$ unit cell in Γ, $k \in \mathbb{T}^d$ and $p \in \mathbb{T}^d_\Gamma$ the first Brillouin zone.

Spectral Analysis

"Fourier" $\mathcal{F} : l^2(\mathbb{Z}^d \times \mathbb{Z}^d \times \Omega; M_{2d}(\mathbb{C}) \to L^2(B_\Gamma \times \mathbb{T}^d \times \mathbb{T}^d_\Gamma \times \Omega; M_{2d}(\mathbb{C}))$

$$\Psi(x, y, \zeta) \mapsto \widehat{\Psi}(x_0, k, p, \zeta) = \sum_{\substack{\xi \in \mathbb{Z}^d \\ \gamma \in \Gamma}} e^{ip\cdot(x-\eta)-ik\cdot\xi} \Psi(x - \xi - \gamma, -\xi, \sigma_\xi \zeta),$$

with $x_0 \in B_\Gamma$ unit cell in Γ, $k \in \mathbb{T}^d$ and $p \in \mathbb{T}^d_\Gamma$ the first Brillouin zone.

Fiber decomposition

$$\boxed{\widehat{M} = \mathcal{F} M \mathcal{F}^{-1} = \int^{\oplus}_{\mathbb{T}^d \times \mathbb{T}_\Gamma{}^d} \widehat{M}(k, p)\, d\tilde{k} d\tilde{p}}$$

with $\widehat{M}(k, p)$ explicit and

$\|\widehat{M}(k, p)\|_{l^2} \leq 1$, for all (k, p), as op. on $l^2(B_\Gamma \times \Omega; M_{2d}(\mathbb{C}))$

Link $\Phi_n(y) \leftrightarrow \widehat{M}^n$

Invant vector $\widehat{\Psi}_1(x,\zeta) = \delta_0(x) \otimes \mathbb{I}$ s.t.

$$(\widehat{M}(0,p)\widehat{\Psi}_1)(x,\zeta) = \widehat{\Psi}_1(x,\zeta), \quad \forall p \in \mathbb{T}^d_\Gamma$$

Link $\Phi_n(y) \leftrightarrow \widehat{M}^n$

Invariant vector $\widehat{\Psi}_1(x,\zeta) = \delta_0(x) \otimes \mathbb{I}$ s.t.

$$(\widehat{M}(0,p)\widehat{\Psi}_1)(x,\zeta) = \widehat{\Psi}_1(x,\zeta), \quad \forall p \in \mathbb{T}^d_\Gamma$$

Characteristic function For any $n \in \mathbb{N}$, $y \in \mathbb{T}^d$,

$$\boxed{\Phi_n(y) = \int_{\mathbb{T}^d_\Gamma} \langle (\delta_0 \otimes \mathbb{I},\ \widehat{M}(y,p))^n \widehat{\rho}_0(y,p)\rangle_{l^2(B_\Gamma \times \Omega; M_{2d}(\mathbb{C}))} \widetilde{dp}}$$

Link $\Phi_n(y) \leftrightarrow \widehat{M}^n$

Invariant vector $\widehat{\Psi}_1(x,\zeta) = \delta_0(x) \otimes \mathbb{I}$ s.t.

$$(\widehat{M}(0,p)\widehat{\Psi}_1)(x,\zeta) = \widehat{\Psi}_1(x,\zeta), \quad \forall p \in \mathbb{T}^d_\Gamma$$

Characteristic function For any $n \in \mathbb{N}$, $y \in \mathbb{T}^d$,

$$\boxed{\Phi_n(y) = \int_{\mathbb{T}^d_\Gamma} \langle (\delta_0 \otimes \mathbb{I}, \ \widehat{M}(y,p))^n \widehat{\rho}_0(y,p) \rangle_{l^2(B_\Gamma \times \Omega; M_{2d}(\mathbb{C}))} \widetilde{d}p}$$

Diffusive/Ballistic Scaling $\Rightarrow \ y \mapsto y/n^\alpha \ll 1$

High Powers

$$\widehat{M}(y/n^\alpha, p)^n \simeq \lambda_1(y/n^\alpha, p)^n P(y/n^\alpha, p)$$

where $\lambda_1(y,p)$ e.v. of largest modulus, $P(y,p)$ spectr. proj.

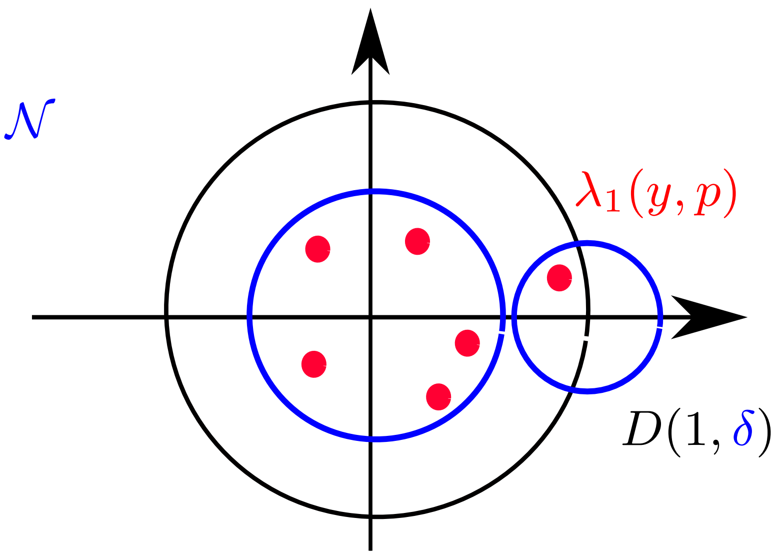
Analytic Perturbation Theory

Assumption S For all $p \in {\mathbb{T}_\Gamma}^d$,

$$\sigma(\widehat{M}(0,p)) \cap \partial D(0,1) = \{1\} \text{ and } 1 \text{ is simple}$$

Consequently

$\exists\, 0 < \delta < 1$, a complex ngbhd. $\mathcal{N}$ of $\{0\} \times {\mathbb{T}_\Gamma}^d$ s.t. $\forall (y,p) \in \mathcal{N}$



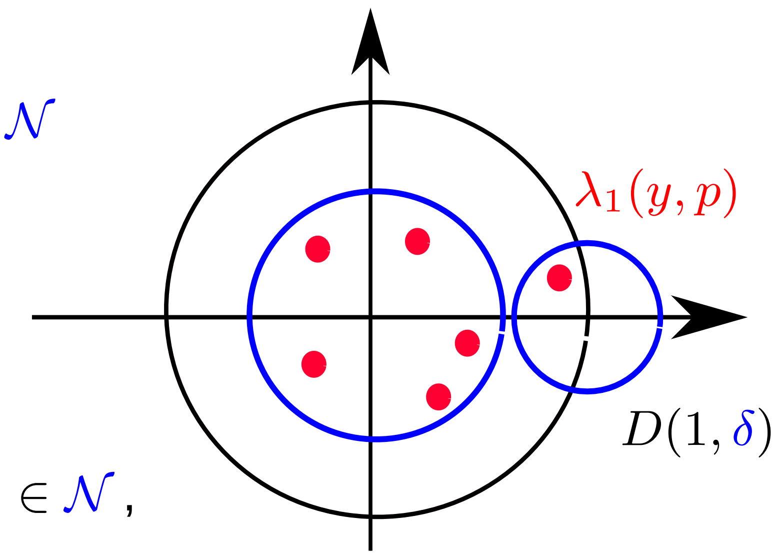
Analytic Perturbation Theory

Assumption S For all $p \in \mathbb{T}_\Gamma{}^d$,

$$\sigma(\widehat{M}(0,p)) \cap \partial D(0,1) = \{1\} \text{ and } 1 \text{ is simple}$$

Consequently

$\exists\, 0 < \delta < 1$, a complex ngbhd. $\mathcal{N}$ of $\{0\} \times \mathbb{T}_\Gamma{}^d$ s.t. $\forall (y,p) \in \mathcal{N}$



Perturbation Theory For $(y,p) \in \mathcal{N}$,

$$\lambda_1(y,p) \equiv 1 + y \frac{i}{2d} \sum_{\tau \in I_\pm} r(\tau) - \frac{1}{2} \langle y | \mathbb{D}(p) y \rangle + O_p(\|y\|^3)$$

The map $p \mapsto \mathbb{D}(p) \in M_d(\mathbb{C})$ is real analytic, and $\mathbb{D}(p) \geq 0$.

Results in Average

Theorem

Under S, unif. in y in cpct sets of $\mathbb{C}^d$,

$$\lim_{n\to\infty} \Phi_n(y/n) = e^{iy\overline{r}}, \qquad \overline{r} = \frac{1}{2d}\sum_{\tau\in I_\pm} r(\tau)$$

$$\lim_{n\to\infty} e^{-in\frac{\overline{r}y}{\sqrt{n}}}\Phi_n(y/\sqrt{n}) = \int_{\mathbb{T}^d_\Gamma} e^{-\frac{1}{2}\langle y|\mathbb{D}(p)y\rangle}\, d\tilde{p}.$$

Results in Average

Theorem

Under S, unif. in y in cpct sets of $\mathbb{C}^d$,

$$\lim_{n\to\infty} \Phi_n(y/n) = e^{iy\overline{r}}, \qquad \overline{r} = \frac{1}{2d} \sum_{\tau \in I_\pm} r(\tau)$$

$$\lim_{n\to\infty} e^{-in\frac{\overline{r}y}{\sqrt{n}}} \Phi_n(y/\sqrt{n}) = \int_{\mathbb{T}^d_\Gamma} e^{-\frac{1}{2}\langle y|\mathbb{D}(p)y\rangle} \, d\tilde{p}.$$

Consequently

$$\lim_{n\to\infty} \frac{\mathbb{E}_\omega \langle X_i \rangle^\omega_{\psi_0}(n)}{n} = \overline{r}_i \qquad \text{Drift}$$

$$\lim_{n\to\infty} \frac{\mathbb{E}_\omega \langle (X - n\overline{r})_i (X - n\overline{r})_j \rangle^\omega_{\psi_0}(n)}{n} = \int_{\mathbb{T}^d_\Gamma} \mathbb{D}_{i\,j}(p) \, d\tilde{p} \qquad \text{Diffusion Matrix}$$

$$\frac{X_n - n\overline{r}}{\sqrt{n}} \to \mathcal{N}(0, \mathbb{D}), \quad \text{if } \mathbb{D}(p) \equiv \mathbb{D} > 0 \qquad \text{CLT}$$

Assumption D

$\mathbb{D}(p) > 0$ and $p \mapsto \langle y | \mathbb{D}(p) y \rangle$ has a non-deg. max $p^*(y)$, in $\mathbb{T}^d_\Gamma$ for any $y \in \mathbb{R}^d \setminus \{0\}$

Rate Funct. For $x \in \mathbb{R}^d$, $\Lambda^*(x) := \sup_{y \in \mathbb{R}^d} \left(\langle y | x \rangle - \frac{1}{2} \langle y | \mathbb{D}(p^*(y)) y \rangle \right) \in [0, \infty]$

Theorem Under D, for all $\Gamma \subset \mathbb{R}^d$, and $0 < \alpha < 1$,

$$\boxed{\lim_{n \to \infty} \frac{1}{n^\alpha} \ln \mathbb{P}((X_n - n\overline{r}) \in n^{(\alpha+1)/2}\,\Gamma) = -\inf_{x \in \Gamma} \Lambda^*(x)}$$

i.o.w.

$$\mathbb{P}((X_n - n\overline{r})/n^{(\alpha+1)/2}) \in \Gamma) \simeq \exp(-n^\alpha \inf_{x \in \Gamma} \Lambda^*(x))$$

Generalizations

- Random Case $e^{-iy\bar{r}\sqrt{n}}\Phi_n^{\omega}(y/\sqrt{n}) \longrightarrow e^{iyX^{\omega}}$, $X^{\omega} \simeq \mathcal{N}(0,\Sigma)$

$$\frac{\langle (X_n^{\omega}-n\bar{r})_i(X_n^{\omega}-n\bar{r})_j\rangle_{\psi_0}}{n} \rightarrow \mathbb{D}_{ij}^{\omega} \simeq (X^{\omega})_i(X^{\omega})_j$$

- Density matrices $\varphi_0 \otimes |0\rangle \mapsto \rho_0 \in \mathcal{B}_1(\mathcal{H})$
- Under stronger assumptions, moderate / large deviations estimates
- Case $\sigma_x \equiv \mathbb{I}$, for all $x \in \mathbb{Z}^d$ dealt with in '11 Ahlbrecht et al & '11 J.
- i.i.d. $C_n^{\omega}(x)$ in space and time can be dealt with
- Random coin op's in space and localization properties for $d \geq 1$

d=1 dealt with in '10 J.-Merkli and '11 Ahlbrecht et al
d>1 dealt with in '12 J.