Localization for Schrödinger operators with Delone potentials

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Schrödinger operators with alloy type potential

Let us consider Schrödinger operators of the form

$$H_D := -\Delta + V_D \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d)$$

- Δ is the d-dimensional Laplacian operator.
- V_D is an alloy-type potential:

$$V_D(x) := \sum_{\zeta \in D} u_\zeta(x), \quad \text{where} \quad u_\zeta(x) = u(x - \zeta)$$

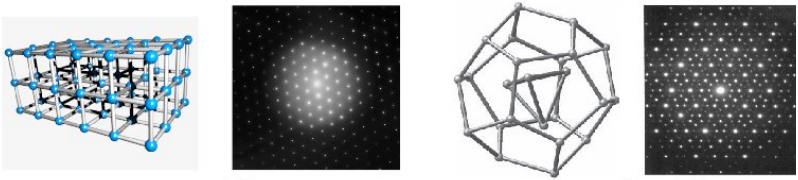
  - The single site potential u is a nonnegative bounded measurable function on $\mathbb{R}^d$ with compact support, uniformly bounded away from zero in a neighborhood of the origin.

It is well-known that if D is a periodic then H_D has ac spectrum. What if D has a more complex structure, like Delone sets ?

Quasicrystals

1984 ('82) D. Shechtman, I. Blech, D. Gratias, J.W. Cahn, *"Metallic phase with long-range orientational order and no translation symmetry"*, Phys. Rev. Letters.

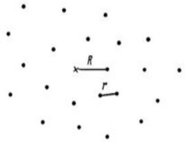
Diffraction patterns



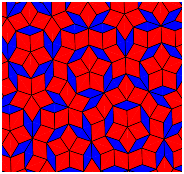
crystal quasicrystal

A ***Delone* set** D of parameters (r, R) is a pure point set in $\mathbb{R}^d$, uniformly discrete (r) and relatively dense (R).

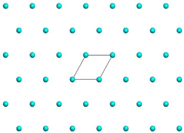
Delone sets



Delone set



Penrose tiling



lattice

Let D a (r, R)-*Delone* set,

$$H(D) = -\Delta + \sum_{\zeta \in D} u(\cdot - \zeta)$$

- The spectrum of $H(D)$ is generically purely singular continuous, within the set of (r, R)-*Delone* sets.

'90, '00's: A. Hof, R. Moody, J.C. Lagarias, B. Solomyak, D. Lenz-P.Stollmann, P. Müller-C. Richard.

QUESTION

Can we get Schrödinger operators with a Delone potential and localization ?

That is: take D a (r,R)-Delone set, $u \geq C\chi_\delta \geq 0$, $\delta < r < R$. Can

$$H_D = -\Delta + \sum_{\zeta \in D} u(\cdot - \zeta)$$

has localization in $[\Sigma_{\inf}, \Sigma_{\inf} + \kappa] \cap \sigma(H_D) \neq 0$, for some $\kappa > 0$, where $\Sigma_{\inf} = \inf \sigma(H_D)$?
There is no randomness here.

- YES!
- How "many" ? A lot from a topological point of view: in progress (dense and union of G_δ)
- How "regular" are they? Rather irregular: for instance, those we construct do not exhibit uniform pattern frequency. There is an infinite number of patterns, repeated an infinite number of times (in progress)

A detour: The Continuous Anderson Hamiltonian

The Anderson Hamiltonian is the random Schrödinger operator

$$H_\omega := -\Delta + V_0 + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d)$$

- Δ is the d-dimensional Laplacian operator.
- V_0 is a bounded periodic background potential.
- V_ω is an alloy-type random potential:

$$V_\omega(x) := \sum_{\zeta \in \mathbb{Z}^d} \omega_\zeta \, u_\zeta(x), \quad \text{where} \quad u_\zeta(x) = u(x - \zeta)$$

  - The single site potential u is a nonnegative bounded measurable function on $\mathbb{R}^d$ with compact support, uniformly bounded away from zero in a neighborhood of the origin.
  - $\omega = \{\omega_\zeta\}_{\zeta \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables, whose common probability distribution μ is non-degenerate with bounded support.

Localization

Theorem (G., Klein 2012)

(Ergodic) Anderson Hamiltonians exhibit a strong form of localization at the bottom of the spectrum without any additional condition on the single site probability distribution.

This strong form of localization holds in an interval

$$[E_{\inf}, E_0] \subset \Sigma \qquad (E_0 > E_{\inf})$$

and includes:

- *Anderson localization: pure point spectrum with exponentially decaying eigenfunctions (with probability one).*
- *Dynamical localization: no spreading of wave packets under the time evolution.*

Moreover, the integrated density of states is Log-Hölder continuous on the interval $[E_{\inf}, E_0]$.

Comments I

- We are only discussing results that hold in arbitrary dimension d. ($d = 1$ is special.)
- If the single-site probability distribution μ has a bounded density (or is Hölder continuous) these results have been known for some time. They also hold for the Anderson model on $\ell^2(\mathbb{Z}^d)$.
- If μ is a Bernoulli distribution, Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) was proved by Bourgain and Kenig (2005).
- Spectral localization (pure point spectrum) for arbitrary μ follows from an extension of the BK results by a Bernoulli decomposition for random variables (Aizenman, G., Klein, Warzel (2009)).
- The proof is based on a multiscale analysis that incorporates the new ideas introduced by Bourgain and Kenig.

Comments II

- Anderson localization was proved for Poisson random potentials by G., Hislop and Klein (2007) using the BK results. The results in this talk, including dynamical localization and log-Hölder continuity of the IDS hold for the Poisson Hamiltonian.
- Related open problems:
  - discrete Bernoulli Anderson model: no UCP
  - Landau Hamiltonian with singular random potential: UCP with exponent 2 instead of $\frac{4}{3}$, which is not enough to perform the MSA
  - singular potential of non definite sign: cannot use the QUCP

Notation

- Given $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, we set

$$\|x\| := \max\{|x_1|, |x_2|, \ldots, |x_d|\} \quad \text{and} \quad \langle x \rangle := \sqrt{1 + \|x\|^2}.$$

- The (open) box of side L centered at $x \in \mathbb{R}^d$:

$$\Lambda_L(x) := \left\{ y \in \mathbb{R}^d;\ \|y - x\| < \tfrac{L}{2} \right\} = x + \left] -\tfrac{L}{2}, \tfrac{L}{2} \right[^d$$

- $\chi_x := \chi_{\Lambda_1(x)}$ is the characteristic function of the unit box centered at $x \in \mathbb{R}^d$.
- Spectral projections:

$$P_\omega(B) := \chi_B(H_\omega) \quad \text{for} \quad B \subset \mathbb{R}^d,$$
$$P_\omega(E) := P_\omega(\{E\}) \quad \text{for} \quad E \in \mathbb{R},$$
$$P_\omega^{(E)} := P_\omega(]-\infty, E]), \quad \text{the Fermi projection with Fermi energy } E.$$

Log-Hölder continuity of the integrated density of states

The integrated density of states: $N(E) := \mathbb{E}\left\{\operatorname{tr} \chi_0 P_\omega^{(E)} \chi_0\right\}$.

Theorem

Let H_ω be an Anderson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$. Then there exists an energy $E_0 > E_{\inf}$, constants C and $\kappa > 0$ such that for all $E_1, E_2 \in [E_{\inf}, E_0]$ with $|E_2 - E_1|$ sufficiently small we have

$$|N(E_2) - N(E_1)| \leq \frac{C}{|\log|E_2 - E_1||^\kappa} .$$

Regular case [Combes, Hislop, Klopp]:
$|N(E_2) - N(E_1)| \leq C\, Q_\mu(|E_2 - E_1|)$, where
$Q_\mu(s) := \sup_{t \in \mathbb{R}} \mu([t, t+s])$ for $s > 0$.

Theorem (Details of Localization) I

Let H_ω be an Anderson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$. Then there exists an energy $E_0 > E_{\inf}$, constants $\beta \in]0,1]$ and $M > 0$, so H_ω exhibits strong localization in the energy interval $[E_{\inf}, E_0]$ in the following sense:

1. Enhanced Anderson localization: The following holds with probability one:
   - H_ω has pure point spectrum in the interval $[E_{\inf}, E_0]$.
   - For all $E \in [E_{\inf}, E_0]$, $\psi \in \operatorname{Ran} P_\omega(E)$, and $\nu > \frac{d}{2}$, we have

     $$\|\chi_x \psi\| \leq C_{\omega,E,\nu} \left\| \langle X \rangle^{-\nu} \psi \right\| e^{-M\|x\|} \qquad \text{for all} \quad x \in \mathbb{R}^d.$$

     In particular, each eigenfunction ψ of H_ω with eigenvalue $E \in [E_{\inf}, E_0]$ is exponentially localized with the non-random rate of decay $m > 0$.
   - The eigenvalues of H_ω in $[E_{\inf}, E_0]$ have finite multiplicity:

     $$\operatorname{tr} P_\omega(E) < \infty \quad \text{for all} \quad E \in [E_{\inf}, E_0].$$

Theorem (Details of Localization) II

2. The following holds with probability one for all $\varepsilon > 0$:
   - Summable uniform decay of eigenfunction correlations (SUDEC):

     For all $E \in [E_{\text{inf}}, E_0]$, $x, y \in \mathbb{R}^d$, and $\nu > \frac{d}{2}$, we have

$$\|\chi_x \phi\| \, \|\chi_y \psi\| \le C_{\omega,\varepsilon,\nu} \left\| \langle X \rangle^{-\nu} \phi \right\| \left\| \langle X \rangle^{-\nu} \psi \right\| \mathrm{e}^{\|x\|^{\frac{1}{2}+\varepsilon}} \mathrm{e}^{-\frac{1}{4} M \|x-y\|^{\frac{\beta}{2}}}$$

     for all $\phi, \psi \in \operatorname{Ran} P_\omega(E)$, and

$$\|\chi_x P_\omega(E)\|_2 \left\|\chi_y P_\omega(E)\right\|_2 \le C_{\omega,\varepsilon,\nu} \left\| \langle X \rangle^{-\nu} P_\omega(E) \right\|_2^2 \mathrm{e}^{\|x\|^{\frac{1}{2}+\varepsilon}} \mathrm{e}^{-\frac{1}{4} M \|x-y|}$$

Theorem (Details of Localization) III

- Semi-uniformly localized eigenfunctions (SULE):

  For all $E \in [E_{\inf}, E_0]$ there exists a "center of localization" $y_{\omega,E} \in \mathbb{R}^d$ for all eigenfunctions with eigenvalue E, in the sense that for all $x \in \mathbb{R}^d$ and $\nu > \frac{d}{2}$ we have

  $$\|\chi_x \phi\| \leq C_{\omega,\varepsilon,\nu} \left\| T_\nu^{-1} \phi \right\| \mathrm{e}^{\left\| y_{\omega,E} \right\|^{\frac{1}{2}+\varepsilon}} \mathrm{e}^{-\frac{1}{4} M \left\| x - y_{\omega,E} \right\|^{\frac{\beta}{2}}} \text{ for } \phi \in \operatorname{Ran} P_\omega(E),$$

  and

  $$\|\chi_x P_\omega(E)\|_2 \leq C_{\omega,\varepsilon,\nu} \left\| T_\nu^{-1} P_\omega(E) \right\|_2 \mathrm{e}^{\left\| y_{\omega,E} \right\|^{\frac{1}{2}+\varepsilon}} \mathrm{e}^{-\frac{1}{4} M \left\| x - y_{\omega,E} \right\|^{\frac{\beta}{2}}}.$$

  Moreover, we have

  $$N_\omega(L) := \sum_{E \in [E_{\inf}, E_0];\, \left\| y_{\omega,E} \right\| \leq L} \operatorname{tr} P_\omega(E) \leq C_{\omega,\varepsilon} L^{(1+2\varepsilon)\frac{d}{\beta}} \quad \text{for} \quad L \geq 1.$$

Theorem (Details of Localization) IV

- Almost sure dynamical localization:

  For all $x, y \in \mathbb{R}^d$ we have

$$\sup_{|f|\leq 1} \left\| \chi_y f(H_\omega) P_\omega([E_{\inf}, E_0]) \chi_x \right\|_1 \leq C_{\omega,\varepsilon} \mathrm{e}^{\|x\|^{\frac{1}{2}+\varepsilon}} \mathrm{e}^{-\frac{1}{4} M \|x-y\|^{\frac{\beta}{2}}} .$$

- Almost sure decay of the Fermi projection:

  For all $E \in [E_{\inf}, E_0]$ and $x, y \in \mathbb{R}^d$ we have

$$\left\| \chi_y P_\omega^{(E)} \chi_x \right\|_1 \leq C_{\omega,\varepsilon} \mathrm{e}^{\|x\|^{\frac{1}{2}+\varepsilon}} \mathrm{e}^{-\frac{1}{4} M \|x-y\|^{\frac{\beta}{2}}} .$$

Theorem (Details of Localization) V

3. Given $b > 0$, for all $s \in \left]0, \frac{\beta}{b+\frac{1}{2}}\right[$ and $x_0 \in \mathbb{R}^d$ we have

- Strong dynamical localization:

$$\mathbb{E}\left\{\sup_{|f|\le 1}\left\|\langle X\rangle^{bd} f(H_\omega)P_\omega([E_{\inf}, E_0])\chi_{x_0}\right\|_1^s\right\} < \infty$$

and

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}}\left\|\langle X\rangle^{bd}\,\mathrm{e}^{-itH_\omega}P_\omega([E_{\inf}, E_0])\chi_{x_0}\right\|_1^s\right\} < \infty.$$

- Strong decay of the Fermi projection:

$$\mathbb{E}\left\{\sup_{E\in[E_{\inf}, E_0]}\left\|\langle X\rangle^{bd} P_\omega^{(E)}\chi_{x_0}\right\|_1^s\right\} < \infty.$$

The Bernoulli-Delone Schrödinger operator

- Let D_1 be a (r, R)-Delone set.
- Take D_2 another (r, R)-Delone
- such that $D_1 \cup D_2$ is a $(\frac{r}{2}, R)$-Delone

(possible: for instance play with the Voronoï diagram associated to D_1).

Consider the Bernoulli-Delone Schrödinger operator

$$H_\omega = -\Delta + \sum_{\zeta \in D_1} u_\zeta + \sum_{\zeta \in D_2} \omega_\zeta u_\zeta$$

with $(\omega_z)_{\zeta \in D_2}$ iid Bernoulli rv.

Write $D_{2,\omega} = \{\zeta \in D_2,\ \omega_\zeta = 1\}$, so that $H_\omega = -\Delta + V_{D_1 \cup D_{2,\omega}}$.
Note that for any given ω, $D_1 \cup D_{2,\omega}$ is a $(\frac{r}{2}, R)$-Delone set.

How to get localization?

APPLY MULTISCALE ANALYSIS

- The multiscale analysis is not sensitive to the geometry of the underlying set where impurities are located (see e.g. [RM12]).
- The multiscale analysis of Bourgain-Kenig for the Bernoulli Schrödinger operator ($D_1 = \emptyset$ and D_2 periodic), applies in a similar way. See [G., Klein 2012] for a detailed version in the ergodic case, with arbitrary non trivial rv.
- But one has to start! GET THE ILSE, that is for E close to the bottom of the spectrum, for some $q \in]\frac{1}{3}, \frac{3}{8}[$,

$$\mathbb{P}(\|\chi_x R_{\omega,\Lambda}(E)\chi_y\| \leq \mathrm{e}^{-m\|x-y\|} \text{ and } \|R_{\omega,\Lambda}(E)\| \leq \mathrm{e}^{L^{1-\varepsilon}}) \geq 1 - L^{-qd},$$

Lifshitz tail ? OK if D_1 and D_2 are periodic.

The case $D_1 = \emptyset$: [G, proc. Qmath10]

ILSE follows easily as in [BK,GKH]. Compare $V_{D_{2,\omega}}$ to an averaged potential $\overline{V} \geq CR^{-d}$ with a good probability, and use the fact that at the bottom of the spectrum $(= 0)$, the kinetic energy is small.

$$\overline{V}_{\omega_\Lambda}(x) := \frac{1}{(KR)^d} \int_{\Lambda_{KR}(0)} \mathrm{d}a\, V_{\omega_\Lambda}(x-a) \geq \frac{c_{u,d}}{R^d} Y_{\omega,\Lambda} \chi_\Lambda(x), \qquad (1)$$

with $K \approx (\log L)^{\frac{1}{d}}$ and

$$Y_{\omega,\Lambda} := \min_{\xi \in \widetilde{\Lambda}} \frac{1}{K^d} \sum_{\zeta \in \Lambda_{K/3}(\xi)} \omega_\zeta \geq \frac{\bar{\mu}}{2}, \qquad (2)$$

with a probility $\geq 1 - \mathrm{e}^{-A_\mu K^d}$, with $\bar{\mu}$ the mean of the probability measure μ, and for some $A_\mu > 0$ (deviation estimate).
We have, for $\varphi \in C_c^\infty(\Lambda)$, $\|\varphi\| = 1$,

$$\begin{aligned}\left\langle \varphi, H_{\omega,\Lambda}\varphi \right\rangle_\Lambda &\geq \langle \varphi, \overline{V}_{\omega_\Lambda}\varphi \rangle + \langle \varphi, (V_{\omega_\Lambda} - \overline{V}_{\omega_\Lambda})\varphi \rangle \\ &\geq \frac{C}{R^d} - cKR\|\nabla_L \varphi\| \geq \frac{C}{R^d} - cKR \left\langle \varphi, H_{\omega,\Lambda}\varphi \right\rangle_\Lambda^{1/2}\end{aligned}$$

and thus $\left\langle \varphi, H_{\omega,\Lambda}\varphi \right\rangle_\Lambda \geq C' R^{-2(d+1)} K^{-2}$.

The case $D_1 = \emptyset$ (end)

[BK,GK] provides localization for $H_\omega = -\Delta + V_{D_{2,\omega}}$, at the bottom of the spectrum, that is in an interval of the type $[0, C_\delta R^{-2(d+1)}(\log R)^{-2}]$, for $R \geq r \geq \delta$, $\delta > 0$ given.

BUT: the sets $D_{2,\omega}$ for which localization is obtained are not Delone anymore (large holes). However, for any $\varepsilon > 0$, for any $x \in \mathbb{R}^d$, for a.e. ω,

$$\lim_{L\to\infty} L^{-(d-\varepsilon)} |\Lambda_L(x) \cap D_{2,\omega}| = +\infty. \tag{3}$$

It does not solve the original problem.

The case $D_1 \neq \emptyset$

PROBLEM: show that for some $\kappa > 0$, with a good enough probability (operators in Λ_L)

$$\inf \sigma(-\Delta_L + V_{D_1} + V_{D_{2,\omega}}) \geq \inf \sigma(-\Delta_L + V_{D_1}) + \kappa.$$

IDEA: pick $K \approx (\log L)^{\frac{1}{d}+\varepsilon}$, divide Λ_L in cubes $\Lambda_K(\gamma_j)$, $j = 1, \cdots, (L/K)^d$, and make sure there is at least one point of $D_{2,\omega}$ in each $\Lambda_K(\gamma_j)$. We have, with $p = \mathbb{P}(\omega_\zeta = 0)$,

$$\mathbb{P}(A_K := \{\omega, \#(\Lambda_K(\gamma_j) \cap D_{2,\omega}) \geq 1, \forall j\}) \geq 1 - (\frac{L}{K})^d p^{c(K/R)^d}.$$

We restrict ourselves to $\omega \in A_K$.
The rest of the argument is deterministic.
We consider the family $H(t) = -\Delta + V_{D_1} + tV_{D_{2,\omega}}$.

Using a QUCP of [RMV12]

We have [RMV12] (operators in Λ_L),

$$\inf \sigma(-\Delta_L + V_{D_1} + tV_{D_2,\omega}) \geq \inf \sigma(-\Delta_L + V_{D_1}) + t\kappa(K),$$

with $\kappa(K) \geq cK^{-K^{4/3}}$ uniformly in ω.
It uses a precise version of Bourgain-Kenig's quantitative unique continuation principe (as in [GK]) combined with a clever decompostion of Λ_L in dominant and non dominant boxes, in order to get a scale free parameter κ.
Next: to start the MSA, we need the size of the gap to be $>> L^{-1}$, that is

$$L \cdot K^{-K^{4/3}} >> 1.$$

Remember $K \approx (\log L)^{\frac{1}{d}+\varepsilon}$. So we need $\frac{4}{3d} < 1$, that is $d \geq 2$.
Case $d = 1$: use Gronwall inequality to improve on the general QUCP. Then $\kappa(K) = c\mathrm{e}^{-cK}$, and the proof applies for p small enough ($p \leq c\mathrm{e}^{-cR}$).

The Quantitative Unique Continuation Principle

Lemma (Bourgain-Kenig, as in G-Klein)

Set $\Lambda = \Lambda_L(x_0)$*. Let* Δ_Λ *be the Dirichlet Laplacian on* $\mathrm{L}^2(\Lambda)$*, let* V *be a bounded potential on* Λ *with* $\|V\|_\infty \le K$*, let* $\Theta \subset \Lambda$ *measurable, and consider* $u \in \mathscr{D}(\Delta_\Lambda)$ *satisfying,*

$$-\Delta_\Lambda u + Vu = 0,$$
$$\left\| u\chi_{\Lambda_\delta(x)\cap\Lambda} \right\| \le Q \quad \textit{for all} \quad x \in \Lambda,$$
$$\|u\chi_\Theta\| \ge \beta \|u\chi_\Lambda\|.$$

Then, there exist finite constants $R_1 > 1$ *and* $M > 0$*, where* R_1 *depends only on* d, K, Q, δ*, and* M *depends only on* d*, such that for all* $x \in \Lambda$ *with*

$$R := \operatorname{dist}(x, \Theta) \ge \max\{R_1, \operatorname{diam}\Theta\} \quad \textit{and} \quad \Lambda_\delta(x) \subset \Lambda,$$

we have

$$\left\| u\chi_{\Lambda_\delta(x)} \right\|^2 \ge R^{-M\left(1+K^{\frac{2}{3}}+\log\beta\right)R^{\frac{4}{3}}} \|u\chi_\Theta\|^2.$$

QUCP for $H_0 = -\Delta + V_0$

Let $H_{0,L} = -\Delta_L + V_{0,L}$ with V_0 bounded, and $E_0 = \inf \sigma(H_0)$.

Theorem (Rojas-Molina - Veselic 2012)

If φ is an eigenfunction of the operator $H_{0,L}$ in an interval I, and D is a Delone set, we have

$$\sum_{\gamma \in D \cap \Lambda_L} \|\varphi\|^2_{B(\gamma,\delta)} \geq C_{UCP}(I,d)\,\|\varphi\|^2_{\Lambda_L}$$

Known with a periodic background: Combes-Hislop-Klopp'03, Combes-Hislop-Klopp'07

i) Application to Wegner estimates

ii) Perturbation of the bottom of the spectrum: denote by $\lambda^L(t) = \inf \sigma(H_{t,L})$ the bottom of the spectrum of $H_{t,L} := -\Delta_L + V_{0,L} + tV_L$ on $\Lambda_L(x)$ with Dirichlet boundary conditions. Then

$$\forall\ t \in (0,1]: \quad \lambda^L(t) \geq \lambda^L(0) + C_{UCP}(u,I,d) \cdot t$$