The Feinberg-Zee random hopping model

Feinberg-Zee model

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Finite or infinite matrices?

$$\begin{pmatrix} a_1 & c_1 & 0 & 0 & 0 & b_1 \\ b_2 & a_2 & c_2 & 0 & 0 & 0 \\ 0 & b_3 & a_3 & c_3 & 0 & 0 \\ 0 & 0 & b_4 & a_4 & c_4 & 0 \\ 0 & 0 & 0 & b_5 & a_5 & c_5 \\ c_6 & 0 & 0 & 0 & b_6 & a_6 \end{pmatrix}$$

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The following pseudospectral phenomenon does not arise for self-adjoint approximations.

If $A_n \to A_\infty$ and $\|(A_n - \lambda I_n)^{-1}\| \to \infty$ as $n \to \infty$ then $\lambda \in \mathrm{Spec}(A_\infty)$, even if λ is not close to the spectrum of any A_n.

The Setting

This is joint work with Simon Chandler-Wilde at Reading.

It follows work about ten years ago by myself and a very recent paper by

Chandler-Wilde, Chonchaiya and Lindner.

The problem is to find the spectrum of a NSA operator on $\ell^2(\mathbf{Z})$ of the form

$$(Af)_n = \sigma_n f_{n-1} + f_{n+1}$$

for all $n \in \mathbf{Z}$.

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We do not consider the corresponding finite problem.

Pseudoergodic matrices, EBD 2001

CW-EBD assumes that $\sigma_n = \pm\sigma$. A is said to be pseudo-ergodic if any finite pattern of $\pm$, such as

$$+ + - - - + - - + + + + + - -+$$

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The probability law governing σ_n is almost irrelevant.

Theorem

All pseudo-ergodic A have the same spectrum. All other B of the same form have $\mathrm{Spec}(B) \subseteq \mathrm{Spec}(A)$.

The strategy

Obtain inner bounds on Spec(A) for a pseudoergodic matrix A by choosing particular matrices B and using

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Obtain outer bounds on $\mathrm{Spec}(A)$ by finding its numerical range and by the use of perturbation arguments.

Chandler-Wilde, Chonchaiya and Lindner

Theorem

If A is pseudo-ergodic and $\sigma_n = \pm 1$ for all $n \in \mathbf{Z}$ then

$$\{z : |z| \leq 1\} \subseteq \mathrm{Spec}(A).$$

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If A is pseudo-ergodic and $\sigma_n = \pm 1$ for all $n \in \mathbf{Z}$ then

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This depends on the use of a 'magic sequence' σ_n that is not pseudo-ergodic. C-W, C, L prove that for every $|\lambda| < 1$ there is a bounded solution of $Af = \lambda f$ if one uses the magic sequence to define A.

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This disproved a conjecture of Feinburg that the spectrum has dimension less than 2.

EBD and Chandler-Wilde, 2011

Theorem

If $0 < \sigma < 1$ *and* $\sigma_n = \pm\sigma$ *for all* n *and* A *is pseudo-ergodic then*

$$\{z : |z| \leq 1\} \backslash H \subseteq \mathrm{Spec}(A)$$

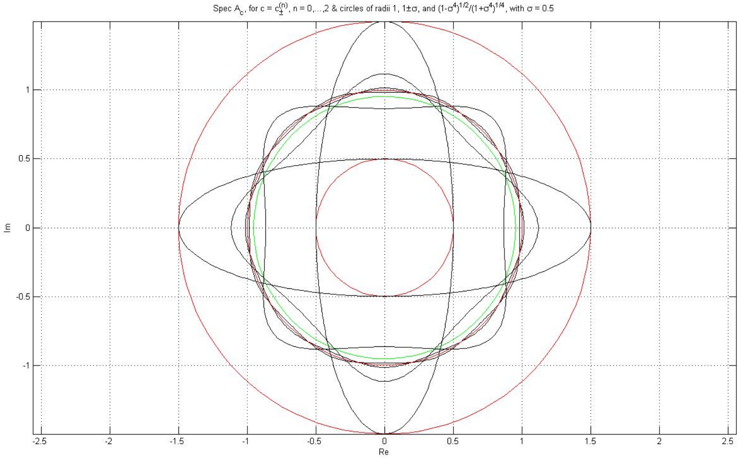
where the hole H *is the intersection of two elliptical regions, namely the interiors of*

$$\frac{x^2}{(1+\sigma)^2} + \frac{y^2}{(1-\sigma)^2} = 1$$

and

$$\frac{x^2}{(1-\sigma)^2} + \frac{y^2}{(1+\sigma)^2} = 1.$$

Some closed curves, sigma=0.5



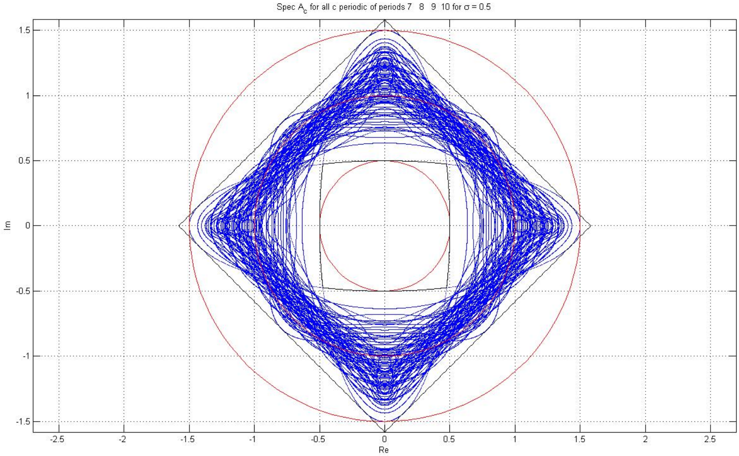
The hole

We cannot prove that the hole is this shape but it certainly contains

$$\{z : |z| < 1 - \sigma\}.$$

Numerical studies suggest we have it right.

All matrices with periods 7 to 10 with sigma=0.5



The first important idea

The operators that we are considering are of the following form.

The Hilbert space $\mathcal{H}$ is the orthogonal direct sum of subspaces $\mathcal{H}_e$ and $\mathcal{H}_o$. If A exchanges these subspaces then it may be written as a 2×2 block matrix.

$$A = \begin{pmatrix} 0 & A_{e,o} \\ A_{o,e} & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} B & 0 \\ 0 & M \end{pmatrix}.$$

Lemma

If $QA = AP$ *then* $PA^2 = A^2 P$, *so* $\mathcal{H}_e$ *and* $\mathcal{H}_o$ *are invariant under the action of* A^2. *If* B *is the restriction of* A^2 *to* $\mathcal{H}_e$ *and* M *is the restriction of* A^2 *to* $\mathcal{H}_o$ *then*

$$\mathrm{Spec}(A^2)\backslash\{0\} = \mathrm{Spec}(B)\backslash\{0\} = \mathrm{Spec}(M)\backslash\{0\}. \quad (1)$$

If A *is invertible then*

$$\mathrm{Spec}(A^2) = \mathrm{Spec}(B) = \mathrm{Spec}(M). \quad (2)$$

Lemma

Given $b \in \Omega$*, let* $c = \Gamma_+(b) \in \Omega$ *be the unique sequence satisfying*

$$c_0 = 1, \qquad c_{2n} + c_{2n+1} = 0, \qquad c_{2n}c_{2n-1} = b_n, \tag{3}$$

for all $n \in \mathbf{Z}$*. Then* A_c^2 *is unitarily equivalent to* $A_b \oplus M_b$ *acting in* $\ell^2(\mathbf{Z}) \oplus \ell^2(\mathbf{Z})$*, where*

$$(M_b f)_n = -f_{n-1} + (c_{2n+1} + c_{2n+2})f_n + f_{n+1} \tag{4}$$

for all $f \in \ell^2(\mathbf{Z})$*. Moreover*

$$\mathrm{Spec}(A_c^2) = \mathrm{Spec}(A_b) = \mathrm{Spec}(M_b).$$

The stable spectrum

This is defined as the union of the essential spectrum $\mathrm{Ess}(A)$ and certain sets $U_n(A)$ for $n \neq 0$.

$\lambda \in \mathrm{Ess}(A)$ if $A - \lambda I$ is not Fredholm.

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$\lambda \in U_n(A)$ if $A - \lambda I$ is Fredholm with index n.

The second important idea

Theorem

Let A correspond to the sequence c_n where c_n has one periodic structure for $n < 0$ and another for $n \geq 0$. Then

$$\mathrm{Ess}(A) \subseteq \mathrm{Stab}(A) \subseteq \mathrm{Spec}(A)$$

and $\mathrm{Stab}(A)$ can be computed in closed form.

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and $\mathrm{Stab}(A)$ can be computed in closed form.

The proof of the final statement uses the fact that the stable spectrum is invariant under compact perturbations of A.

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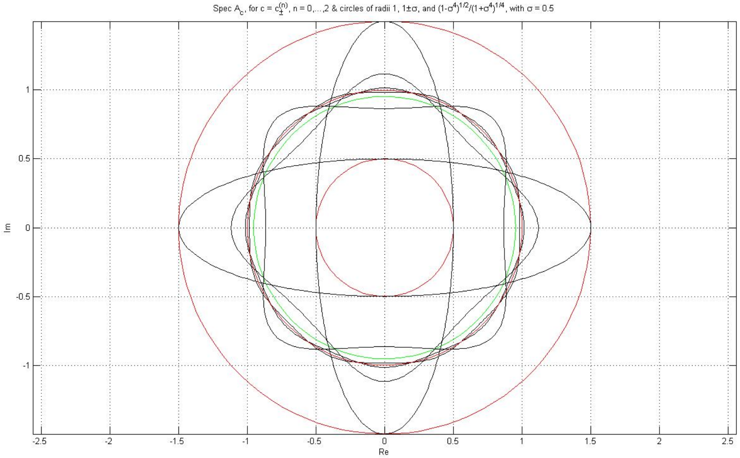

Summary

- With great effort we have found a large part of the spectrum of the infinite tridiagonal matrix

$$(Af)_n = \pm\sigma f_{n-1} + f_{n+1}$$

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- Even for this example, there remains more to be done, and it needs new ideas.
- What is lacking is a systematic method of approaching all such problems, and perhaps this does not exist.
- But perhaps it does!